ADOMIAN’S DECOMPOSITION METHOD FOR SOLVING ANHARMONIC OSCILLATOR SYSTEMS

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ABSTRACT-The anharmonic oscillator is solved quickly, easily, and elegantly by Adomian’s methods for solution of nonlinear stochastic differential equations emphasizing its applicability to nonlinear deterministic equations as well as stochastic equations.

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Previous investigations of fluctuations of dynamical variables for nonlinear oscillating systems have been limited to approximation methods generally involving a linearization replacing the actual nonlinear system with a so-called equivalent linear system and in the case of stochasticity, with averaging which are generally nonvalid and assumption of nonphysical white noise processes for reasons of mathematical tractability. The anharmonic oscillator, the Duffing oscillator, and others are easily dealt with by the methods developed by G. Adomian and his students [6]. We have emphasized the anharmonic oscillator for nonlinear stochastic case to stress the point that the methods developed for nonlinear stochastic systems are valid in the linear or deterministic limiting cases as well and can be compared with known results there.[5-6]
Methods developed by G. Adomian[6] and his co-workers solve linear stochastic differential equations of the form.

\[ Ly + Ny = x(t) \]

Where \( x \) may be a stochastic process. Term \( Ny \) is nonlinear, possibly involving coefficients of differential terms and \( L \) is a linear stochastic differential operator of order \( n \) which can decompose into deterministic and random parts \( L+R \). Operator \( L \) may be taken as the highest order of \( L \), but not necessary. Instead one may choose \( L \) to get the simplest Green’s function. When we have no randomness, we can let \( R \) represent the additional deterministic part, so \( L \) may be taken as the highest ordered term alone.

Determining \( L^{-1} \), we can write

\[ y = L^{-1}x - L^{-1}Ry - L^{-1}Ny. \]

More generally, if \( f_y = Ly + Ry + Ny = x \). The method basically assumes the solution can be written \( y = f^{-1}x \) and assumes a decomposition for the solution into component \( y_0 + y_1 + \ldots \) with \( y_0 = L^{-1}x \), if initial conditions are zero.

Adomian[1]. Solution of the homogeneous equation \( Ly=0 \) must be added to \( L^{-1}x \) otherwise.[7]

ANALYSIS
The anharmonic oscillator is given by

\[ \frac{d^2u}{dt^2} + k^2 \sin u = 0 \]

\( k^2 = (g/l) \) for large amplitude motion.

Let \( L=d^2/dt^2 \) and \( Nu=k^2 \sin u \) so that eqn. (1) can be written as \( Lu + Nu = 0 \). We can do it as well for \( \gamma \neq 0 \) and \( u'(0) = \omega \neq 0 \).

Equation (2) is a nonlinear deterministic homogeneous differential equation. Assume initial conditions \( u(0) = \gamma = 0 \) and \( u'(0) = \omega \neq 0 \). Adomian [1],[4] have done it for \( u(0) = \gamma = \text{const} \) and \( u'(0) = 0 \). We can do it as well for \( \gamma \neq 0 \) and \( u'(0) = \omega \neq 0 \).

Equation (2) in the form \( Ly+Nu = x \) considered by Adomian as a special case of nonlinear stochastic differential operator equations with no stochastic terms present.thus, \( Lu = x-Nu \) or \( u = L^{-1}x - L^{-1}Nu \), we get the Adomian decomposition as a special case of nonlinear stochastic differential operator equations with no stochastic terms present.

\[ u(t) = u(0) + u_1(0) - L^{-1}\int_0^t A_n \]

since \( x=0 \).

\[ Nu \text{ term becomes } \sum_{n=0}^\infty A_n \text{ where } A_n \text{ is called Adomian's polynomial given as } A_n = \sum_{i=1}^{n} c(i,n) h_i (u_0) \]

(4)

Where for any \( n \) we compute \( h_i(u_0), i = 1, 2, \ldots n \) by differentiating \( f(u) \) \( i \) times with respect to \( u \). for \( c(i,n), i \) is the number of integers whose sum is equal to \( n \).[8]

\[ h_i(u_0) = \frac{d^i}{du^i} f(u_0) \]

(5)

Where \( u(0)=0, u'(0)=\omega \) from the initial condition, equation (3) becomes

\[ u(t) = t \omega - L^{-1} \sum_{n=0}^\infty A_n \]

Take
The solution of the equation is

\[ u(t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots \]

with

\[ u_1 = -L^{-1}A_0, u_2 = -L^{-1}A_1, \ldots, u_n = -L^{-1}A_{n-1} \text{ and} \]

\[ \ldots (7) \]

if \( u = \sin u \).

From equation (4), we obtain

\[ A_0 = \sin u_0 = \sin t \text{ (from eqn. 6)} \]

\[ A_1 = c(1 \ 1) h_1(u_0) = u_1 \cos u_0 \]

\[ A_2 = c(1 \ 2) h_1(u_0) + c(2 \ 2) h_2(u_0) = u_2 \cos u_0 + \frac{u_2}{3!}(\sin u_0) \]

\[ \ldots (8) \]

\[ A_3 = -\frac{u_2^3}{3!}\cos u_0 - u_1 \sin u_0 + u_3 \cos u_0 - u_1 \sin u_0 + u_3 \cos u_0 \]

\[ \ldots (9) \]

\[ A_4 = -\frac{u_2^4}{4!}\cos u_0 - u_1 \sin u_0 + u_3 \cos u_0 - u_1 \sin u_0 + u_3 \cos u_0 \]

\[ \ldots (10) \]

Using the well-known properties of trigonometric function, we have

\[ u_0^2 = \frac{\sin^2 u_0}{3!}, \sin^2 u_0 = \frac{\sin^2 u_0 + \cos^2 u_0}{2} \]

\[ = \sin^2 u_0 + \frac{2}{3!} \sin 2u_0 \cos 2u_0 \]

\[ \ldots (11) \]

Thus,\n
\[ \sin u = \sin u_0 \cos u_0 + \frac{1}{2!} \sin 2u_0 \cos 2u_0 + \sin u_0 \sin 2u_0 \cos 2u_0 \]

For a small angular velocity the value of \( \sin 2u_0 \cos 2u_0 \) is almost zero and the equation will be satisfied. If however, the angular velocity is large, then by virtue of the increasing integral power of \( \omega \) and the diminishing value of \( \omega t \), the equation will identically vanish to zero. Hence, we can see even from only a two term series that our solution will satisfy the original equation at least when \( \omega t \) is small.
\[ \phi_n = u_0 - L^{-1} \sum_{i=0}^{n-1} A_i, \quad \phi_{n+1} = u_0 - L^{-1} \sum_{i=0}^{n} A_i = u_0 - L^{-1} \sum_{i=0}^{n-1} A_i - L^{-1} A_n \]

| \phi_{n+1} - \phi_n | = |L^{-1} A_n| = |u_{n+1}| \rightarrow 0

Examination of the components \( u_n \) term by term shows convergence [2]. To verify solutions with \( u_n = \sum_{i=0}^{n-1} u_i \), substitute into \( Lu + Nu = 0 \)

\[ \sum_{i=0}^{n-1} A_i = 0 \]

Thus, approximating the derivative term to \( u_1 \), for example, by writing \( \phi_2 = u_0 + u_1 \), the \( \sum u_i \) would include only \( A_0 \) since \( u_1 = -L^{-1} A_0 \) and depends only on \( A_0 \).

By solving the equation \( Lu + Nu = 0 \) using the approximation. \( \phi_n = \sum_{i=0}^{n-1} u_i \), for \( u \), we have \( \phi_n = -L^{-1} \sum A_i \) but

\[ \lim_{n \to \infty} \phi_n = u, \quad \lim_{n \to \infty} \sum A_i = Nu. \]

So the equation is satisfied. Existence and uniqueness are guaranteed by G. Adomian, stochastic systems [4] and [9] and for the convergent of the series solution[2],[3]

**Conclusions**

In this paper, ADM has been successfully applied to finding the solutions of anharmonic oscillator equation. The results show that the Adomian decomposition method is a powerful mathematical tool for finding the analytical solutions of linear and nonlinear partial differential equations, and therefore, can be widely applied in engineering problems.

**REFERENCES**


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