ADOMIAN'S DECOMPOSITION METHOD FOR SOLVING ANHARMONIC OSCILLATOR SYSTEMS

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ABSTRACT-The anharmonic oscillator is solved quickly, easily, and elegantly by Adomian's methods for solution of nonlinear stochastic differential equations emphasizing its applicability to nonlinear deterministic equations as well as stochastic equations.

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Previous investigations of fluctuations of dynamical variables for nonlinear oscillating systems have been limited to approximation methods generally involving a linearization replacing the actual nonlinear system with a so-called equivalent linear system and in the case of stochasticity, with averaging which are generally nonvalid and assumption of nonphysical white noise processes for reasons of mathematical tractability. The anharmonic oscillator, the Duffing oscillator, and others are easily dealt with by the methods developed by G. Adomian and his students [6]. We have emphasized the anharmonic oscillator for nonlinear stochastic case to stress the point that the methods developed for nonlinear stochastic systems are valid in the linear or deterministic limiting cases as well and can be compared with known results there.[5]-[6].

Methods developed by G. Adomian[6] and his co-workers solve linear stochastic differential equations of the form.

$$Ly + Ny = x(t)$$

Where x may be a stochastic process. Term Ny is nonlinear, possibly involving stochastic coefficients of differential terms

and L is a linear stochastic differential operator of order n which can decompose into deterministic and random parts L+R. Operator L may be taken as the highest order of L., but not necessary. Instead one may choose L to get the simplest Green's function. When we have no randomness, we can let R represent the additional deterministic part, so L may be taken as the highest ordered term alone.

Determining L^{-1}, we can write

$$y = L^{-1}x - L^{-1}Ry - L^{-1}Ny.$$

More generally, if: $fy = Ly+Ry+Ny = x$.

The method basically assumes the solution can be written

$y = f^{-1}x$

and assumes a decomposition for the solution into component $y_0 + y_1 + \ldots$ with $y_0 = L^{-1}x$, if initial conditions are zero, Adomian[1]. Solution of the homogeneous equation Ly=0 must be added to $L^{-1}x$ otherwise.[7]

ANALYSIS

The anharmonic oscillator is given by

$$\frac{d^2u}{dt^2} + k^2 \sin u = 0 \qquad \ldots(1)$$

$k^2 = (g/l)$ for large amplitude motion.

Let $L = d^2/dt^2$ and $Nu = k^2 \sin u$ so that eqn. (1) can be written as

$Lu + Nu = 0$, Nu is the nonlinear part. …(2)

Equation (2) is a nonlinear deterministic homogeneous differential equation. Assume initial conditions $u(0) = \gamma = 0$ and $u^1(0) = \omega \neq 0$. Adomian [1],[4] have done it for $u(0) = \gamma = \text{const}$ and $u^1(0) = 0$. We can do it as well for $\gamma \neq 0$ and $u^1(0) = \omega \neq 0$

Equation (2) is in the form Ly+Ny=x considered by Adomian as a special case of nonlinear stochastic differential operator equations with no stochastic terms present.thus,

$Lu = x - Nu$ or $u = L^{-1}x - L^{-1}Nu$,

Which we assume the Adomian decomposition $\sum_{n=0}^{n} u_n$ with $u_o = L^{-1}x$ plus the homogeneous solution. Operating on the integral equation. (2) for u with L^{-1}. We obtain

$L^{-1}Lu = L^{-1}x - L^{-1}Nu$,

So that,

$$u(t) = u(0) + tu^1(0) - L^{-1}\sum_{n=0}^{\infty} A_n \qquad \ldots(3)$$

since x=0;

Nu term becomes $\sum_{n=0}^{\infty} A_n$, *where* A_n *is called Adomian's polynominal given as* $A_n = \sum_{i=1}^{\infty} c(i,n) h_1(u_0)$ …(4)

Where for any n we compute $h_i(u_o)$, i= 1,2, … n by differentiating f(u) i times with respect to u. for c(i,n), i is the number of integers whose sum is equal to n.[8]

$$h_i(u_0) = \frac{d^i}{du^i} f(u(\lambda))/_{\lambda=0} \qquad \ldots(5)$$

Where u(0)=0, $u^1(0) = \omega$ from the initial condition, equation (3) becomes

$$u(t) = t\,\omega - L^{-1}\sum_{n=0}^{\infty} A_n$$

Take

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$u_0=u(0) + tu^1(0)= t\omega$ . . .(6)

The solution of the equation is

$$u(t)=\sum_{n=0}^{\infty} u_n = u_0+u_1+u_2+\cdots$$

with

$u_1 = -L^{-1}A_0, u_2 = -L^{-1}A_1 \ldots, u_n = -L^{-1}A_{n-1}$ and . . .(7)

f(u) = sin u.

From equation (4), we obtain

$A_0=\sin u_0=\sin t\omega$ (from eqn. 6)

$A_1=c(1\ 1)\ h_1(u_0)= u_1\cos u_0$

$A_2=c(1\ 2)\ h_1(u_0)+ c(2\ 2)\ h_2(u_0)= u_2\cos u_0+\frac{U_1^2}{2!}(-\sin u_0)$

$= - \frac{U_1^2}{2!}\sin u_0+ u_2\cos u_0$

$A_3=c(1\ 3)\ h_1(u_0)+ c(2\ 3)\ h_2(u_0))+ c(3\ 3)\ h_3(u_0)$

$= u_3\cos u_0-u_1u_2\sin u_0-\frac{U_1^3}{3!}\cos u_0$

i.e., $A_3= - \frac{U_1^3}{3!}\cos u_0 -u_1u_2\sin u_0+u_3\cos u_0$

$A_4=-\frac{U_1^4}{4!}\sin u_0 -\frac{U_2^2}{2!}\sin u_0 -\frac{U_1^2}{2!} u_2 \cos u_0- u_1u_3\sin u_0+u_4\cos u_0$

Equation (7) therefore becomes

$u(t)= = t\omega - L^{-1}[A_0+A_1+\cdots] = u_0+u_1+u_2+\cdots$

With

$u_1=\int dt\int dt\ k^2\sin u_0=-\int dt\int dt\ k^2\sin t\omega$

$= - k^2(-\sin t\omega/\omega^2)$

i.e. $u_1= \frac{k^2}{\omega^2} \sin t\omega$ . . . (8)

$u_2=-L^{-1}k^2[u_1 \cos t\omega]$

$=-k^2L^{-1}[k^2/\omega^2 \sin t\omega\cos t\omega]$

$= -k^2.k^2/\omega^2L^{-1}[\sin t\omega\cos t\omega]$

from which

$u_2 = -k^2k^2/\omega^2L^{-1}[\frac{1}{2}\sin 2t\omega]$

$= -k^4/\omega^2\frac{1}{2}[-\sin 2t\omega/4\omega^2]$

$u_2=1/8(k/\omega)^4\sin 2t\omega$ . . .(9)

$u_3= -L^{-1}A_2 = -L^{-1}k^2\left[-\frac{U_1^2}{2}\sin u_0+u_2\cos u_0\right]$

$=-L^{-1}k^2\left\{-\frac{1}{2}\left(\frac{k^2}{\omega^2}\sin t\omega\right)^2\sin t\omega + \frac{1k^4}{8\omega^4}\sin 2t\omega\cos t\omega\right\}$

$=-k^2L^{-1}\left\{-\frac{1}{2}\frac{k^4}{\omega^4}\sin^3 t\omega\right\}-k^2L^{-1}\left\{\frac{1}{8}\frac{k^4}{\omega^4}\sin 2t\omega\cos t\omega\right\}$

$=\frac{1}{2}\frac{k^6}{\omega^4}L^{-1}(\sin^3 t\omega) - \frac{1}{8}\frac{k^6}{\omega^4}L^{-1}(\sin 2t\omega\cos t\omega)$

Using the well-known properties of trigonometric function, we have

$\sin^3 t\omega = ¼((3\sin t\omega)-\sin^3 t\omega)$

$\sin^2 t\omega\cos t\omega = ½(3\sin^3 t\omega+\sin t\omega)$

We have

$u_3=$

$\frac{k^6}{2\omega^4}L^{-1}¼(3\sin t\omega + \sin 3t\omega) - \frac{k^6}{8\omega^4}L^{-1}½(\sin 3t\omega + \sin t\omega)$

$=$

$\frac{k^6}{2\omega^4}.¼\left(-\frac{3}{\omega^4}\sin t\omega + \frac{1}{3^2\omega^2}\sin 3t\omega\right) - \frac{k^6}{8\omega^4}.½\left(-\frac{1}{3^2w^2}\sin 3t\omega - \frac{1}{\omega^2}\sin t\omega\right)$

$= \frac{k^6}{8\omega^6}\left(-3\sin t\omega + \frac{1}{3^2}\sin 3t\omega\right) + \frac{k^6}{16\omega^6}\left(\frac{1}{3^2}\sin 3t\omega + \sin t\omega\right)$

$=\frac{k^6}{8\omega^6}\left\{-3\sin t\omega + \frac{1}{2}\sin t\omega + \frac{1}{3^2}(\sin 3t\omega + ½\sin 3t\omega)\right\}$

$=\frac{k^6}{8\omega^6}\left\{-5/2\sin t\omega + \frac{1}{2}\sin t\omega + \frac{1}{3^2}.\frac{3}{2}\sin 3t\omega\right\}$

$\therefore\ u_3= \frac{k^6}{16\omega^6}\left\{-5\sin t\omega + \frac{1}{3}\sin 3t\omega\right\}$ . . . (10)

$u_4=-L^{-1}A_3$

With

$A_3=-\frac{U_1^3}{3!}\cos u_0 -u_1u_2\sin u_0+u_3\cos u_0$

$= -\frac{1}{3!}\left(\frac{k^2}{w^2}\sin t\omega\right)^3\cos t\omega - \left(\frac{k^2}{w^2}\sin t\omega\right)\left(\frac{1}{8}\frac{k^4}{\omega^4}\sin^2 t\omega\right)(\sin t\omega)$

$+\frac{k^6}{16\omega^6}(-5\sin t\omega + ⅓\sin 3t\omega)(\cos t\omega)$

$= -\frac{1}{3!}\frac{k^6}{\omega^6}\sin^3 t\omega\cos t\omega - \frac{1}{8}\frac{k^6}{\omega^6}\sin^2 t\omega\sin 2tw$

$+\frac{1}{16}\frac{k^6}{\omega^6}\left(-5\sin t\omega\cos t\omega + \frac{1}{3}\sin 3t\omega\cos t\omega\right)$

$-\frac{1}{6}\frac{k^6}{\omega^6}\left\{\frac{3}{4}\sin t\omega\cos t\omega - ¼\sin 3t\omega\cos t\omega\right\} - \frac{1}{8}\frac{k^6}{\omega^6}\{½\sin 2 - ¼\sin 4t\omega\}$

$+\frac{1}{16}\frac{k^6}{\omega^6}\left\{-\frac{5}{2}\sin 2t\omega + \frac{1}{6}\sin 2t\omega + \frac{1}{6}\sin 4t\omega\right\}$

$= -\frac{1}{6}\frac{k^6}{\omega^6}\left\{\frac{1}{4}\sin 2t\omega - \frac{1}{8}\sin 4t\omega\right\}$

$+\frac{1}{16}\frac{k^6}{\omega^6}\left\{-\sin 2t\omega - \frac{14}{6}\sin 2t\omega + \frac{1}{2}\sin 4t\omega + \frac{1}{6}\sin 4t\omega\right\}$

$= -\frac{1}{16}\frac{k^6}{\omega^6}\left\{\frac{4}{6}\sin 2t\omega - \frac{4}{12}\sin 4t\omega\right\} + \frac{1}{16}\frac{k^6}{\omega^6}\left\{-\frac{20}{6}\sin 2t\omega + \frac{4}{6}\sin 4t\omega\right\}$

$=\frac{1}{16}\frac{k^6}{\omega^6}\left\{-\frac{4}{6}\sin 2t\omega + \frac{4}{12}\sin 4t\omega\right\} - \frac{20}{6}\sin 2t\omega + \frac{4}{6}\sin 4t\omega\}$

$=\frac{1}{16}\frac{k^6}{\omega^6}\left\{-\frac{24}{6}\sin 2t\omega + \sin 4t\omega\right\}$

i.e.

$A_3 = \frac{1}{16}\frac{k^6}{\omega^6}(-4\sin 2t\omega + \sin 4t\omega)$

So that,

$u_4=-L^{-1}k^2\left(\frac{1}{16}\frac{k^6}{\omega^6}\right)(-4\sin 2t\omega + \sin 4t\omega)$

$=-k^2\left(\frac{1}{16}\frac{k^6}{\omega^6}\right)L^{-1}(-4\sin 2t\omega + \sin 4t\omega)$

$=-k^2\left(\frac{1}{16}\frac{k^6}{\omega^6}\right)\left\{-4\left(-\frac{1}{2^2\omega^2}\sin 2t\omega\right) + \frac{1}{4^2\omega^2}(\sin 4t\omega)\right\}$

$= -k^2\left(\frac{1}{16}\frac{k^6}{\omega^6}\right)\left\{\frac{1}{\omega^2}\sin 2t\omega - \frac{1}{4^2\omega^2}\sin 4t\omega\right\}$

$=\left(\frac{1}{16}\frac{k^6}{\omega^6}\right)\left\{-\sin 2t\omega + \frac{1}{16}\sin 4t\omega\right\}$

$u_4= \frac{1}{64}\frac{k^6}{\omega^6}\left(-4\sin 2t\omega + \frac{1}{4}\sin 4t\omega\right)$ . . . (11)

:

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Thus,

$u(t)= t\omega + \frac{k^2}{\omega^2}\sin t\omega + \frac{1}{8}\frac{k^4}{\omega^4}\sin 2t\omega$

$\sin u= \sin t\omega\cos\left(\frac{k^2}{\omega^2}\sin t\omega\right) + \sin\left[\frac{k^2}{\omega^2}\sin t\omega\right]\cos t\omega$

$\frac{du}{dt} = \omega + \frac{k^2}{w}\cos t\omega \Rightarrow \frac{d^2u}{dt^2} = -k^2\sin t\omega$

Substitute back into equation (1) to obtain

$-k^2\sin t\omega + k^2\left[\sin t\omega\cos\left(\frac{k^2}{\omega^2}\sin t\omega\right) + \sin\left(\frac{k^2}{\omega^2}\sin t\omega\right)\cos t\omega\right]$

$= -k^2\sin t\omega + k^2\sin t\omega\cos\left(\frac{k^2}{\omega^2}\sin t\omega\right) +$

$k^2\cos t\omega\sin\left(\frac{k^2}{\omega^2}\sin t\omega\right)$

For a small angular velocity the value of $\sin t\omega$ is almost zero and the equation will be satisfied. If however, the angular velocity is large, then by virtue of the increasing integral power of ω and the diminishing value of $\cos t\omega$, the equation will identically vanish to zero. Hence, we can see even from only a two term series that our solution will satisfy the original equation at least when ωt is small.

The n-term approximation ϕ_n is given by

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$\phi_n = u_0 - L^{-1}\sum_{i-0}^{n-1} A_i$, $\phi_{n+1} = u_0 - L^{-1}\sum_{i-0}^{n} A_i = u_0 - L^{-1}\sum_{i-0}^{n-1} A_i - L^{-1}A_n$

$|\phi_{n-1}-\phi_n| = |L^{-1}A_n| = |u_{n+1}| \to 0$

Examination of the components u_n term by term shows convergent [2]. To verify solutions with $u_n = \sum_{i-0}^{n-1} u_i$ substitute into $Lu + Nu = 0$

$$L\phi_{n+1} + \sum_{i=0}^{n-1} A_i = 0$$

Thus, approximating the derivative term to u_1, for example, by writing

$\phi_2 = u_o + u_1$, the $\sum u_i$ would include only A_0 since $u_1 = -L^{-1}A_0$ and depends only on A_0.

By solving the equation Lu+Nu=0 using the approximation. $\phi_n = \sum_{i=0}^{n-1} u_i$, for u, we have $\phi_n = -L^{-1}\sum A_n$ but $\lim_{n\to\infty}\phi_n = u$, $\lim_{n\to\infty}\sum A_n = Nu$,

So the equation is satisfied. Existence and uniqueness are guaranteed by G. Adomian, stochastic systems [4] and [9] and for the convergent of the series solution[2],[3]

Conclusions

In this paper, ADM has been successfully applied to finding the solutions of anharmoic oscillator equation. The results show that the Adomian decomposition method is a powerful mathematical tool for finding the analytical solutions of linear and nonlinear partial differential equations, and therefore, can be widely applied in engineering problems.

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