Solution of Differential and Integral Equations Using Fixed Point Theory

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Abstract: In this paper we have introduced the concept of Fixed point Theory to solve the Differential and Integral Equations. Using fixed point theory existence and uniqueness of solution of differential and integral equation can be verified.

Keywords: Contraction, Differential Equations, Fixed Points, Integral Equations.

I. INTRODUCTION

Over the past few decades, fixed point theory of Lipschitzian mappings has been developed into an important field of study in both pure and applied mathematics. The main purpose of this paper is to present some of the basic techniques and results of this theory.

Fixed point theory concerns itself with a very simple, and basic, mathematical setting. For a function f that has a set X as both domain and range, a fixed point of f is a point x of X for which $f(x) = x$. Two fundamental theorems concerning fixed points are those of Banach and of Brouwer. In Banach's theorem, X is a complete metric space with metric d and $f: X \rightarrow X$ is required to be a contraction, that is, there must exist $L < 1$ such that $d(f(x), f(y)) \leq Ld(x, y)$ for all $x, y \in X$. The conclusion is that f has a fixed point, in fact exactly one of them.

Brouwer's theorem requires X to be the closed unit ball in a Euclidean space and $f: X \rightarrow X$ to be a map, that is, a continuous function. Again we can conclude that f has a fixed point. But in this case the set of fixed points need not be a single point, in fact every closed nonempty subset of the unit ball is the fixed point set for some map. The metric on X in Banach's theorem is used in the crucial hypothesis about the function, that it is a contraction.

1 Fixed Point Theorems

Fixed points are of interest in themselves but they also provide a way to establish the existence of a solution to a set of equations. For example, in theoretical economics, such as

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general equilibrium theory, there comes at point where one needs to know whether the solution to a system of equations necessarily exists or more precisely under which conditions will a solution necessarily exists. The mathematical analysis of this question usually relies on fixed point theorems.

2 Lipschitz Mapping

In 1922 Banach published his fixed point theorem also known as Banach's Contraction Principle uses the concept of Lipschitz mapping.

Definition 2.1: Let (X, d) be a metric space. The map $T: X \rightarrow X$ is said to be Lipschitzian if there exists a constant $\sigma(T) > 0$ (called Lipschitz constant) such that

$$d\big(T(x), T(y)\big) \leq \sigma(T) d(x, y) \; \forall \; x, y \in X \; .$$

Definition 2.2: A Lipschitzian mapping with Lipschitzian constant $\sigma(T) < 1$ is called contraction.

Theorem 2.1[2]: (Banach Contraction Principle)

Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a contraction mapping. Then T has a unique fixed point x_0 and for each $x \in X$ we have

$$\lim_{n \rightarrow \infty} T^n(x) = x_0$$

Moreover for each $x \in X$, we have

$$d(T^n(x), x_0) \leq \frac{\big(\sigma(T)\big)^n}{1 - \sigma(T)} d(T(x), x)$$

3 Fixed Points

Definition 3.1: Let $< X, d >$ be a metric space and $T: M \subset X \rightarrow X$ be a map. A solution of $Tx = x$ is called a fixed point of T.

Example[3]: Find a zero of map $F: M \subset X \rightarrow X$. This problem can be formulated in different ways as a fixed point problem.
e.g.

$i)\; Tx = x - F(x)$

$ii)\; Tx = x - wF(x), \;\; w \in R \;\; (Linear \;\; relaxation)$

$iii)\; Tx = x - \big(DF(x)\big)^{-1} F(x) \; (Newton's \; Method)$

4 Differential Equations

Let $f(x, y)$ be a continuous real-valued function on $[a, b] \times [c, d]$. The Cauchy initial value problem is to find a continuous differentiable function y on $[a, b]$ satisfying the differential equation

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$$\frac{dy}{dx} = f(x\,,y)\,,\ y(x_0) = y_0 \tag{4.1}$$

Consider the Banach space $C[a\,,b]$ of continuous real-valued functions with supremum norm defined by $\|y\| = \sup\{|y(x)| : x \in [a\,,b]\}$.
Integrating (4.1) , we obtain an integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t\,,y(t))\,dt \tag{4.2}$$

The problem (4.1) is equivalent the problem solving the integral equation (4.2).
We define an integral operator $T: C[a\,,b] \to C[a\,,b]$ by

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t\,,y(t))\,dt$$

Thus, a solution of Cauchy initial value problem (4.1) corresponds with a fixed point of *T*. One may easily check that if *T* is contraction, then the problem (4.1) has a unique solution.

Now our purpose is to impose certain conditions on *f* under which the integral operator *T* is Lipschitzian with $\sigma(T) < 1$.

Theorem 4.1[1]: Let $f(x\,,y)$ be a continuous function of $Dom(f) = [a\,,b] \times [c\,,d]$ such that f is Lipschitzian with respect to *y*, that is there exists $L > 0$ such that

$$|f(x\,,u) - f(x\,,v)| \leq L|u - v| \ for\ all\ u\,,v \in [c\,,d] and\ for\ \ x \in [a\,,b]$$

Suppose $(x_0\,,y_0) \in int(Dom(f))$. Then for sufficiently small $h > o$, there exists a unique solution of the problem (4.1).

Proof: Let $M = \sup\{|f(x\,,y)| : x\,,y \in Dom(f)\}$ and choose $h > 0$ such that $Lh < 1$ and $[x_0 - h\,,x_0 + h] \subseteq [a\,,b]$.
Set $C \coloneqq \{y \in C[x_0 - h\,,x_0 + h]: |y(x) - y_0| \leq Mh\}$.
Then C is a closed subset of the complete metric space $C[x_0 - h\,,x_0 + h]$ and hence C is complete. Note $T: C \to C$ is a contraction mapping. Indeed, for $x \in [x_0 - h\,,x_0 + h]$ and two continuous functions $y_1\,,y_2 \in C$, we have

$$\begin{aligned} \|Ty_1 - Ty_2\| &= \left\| \int_{x_0}^{x} f(x\,,y_1) - f(x\,,y_2)dt \right\| \\ &\leq\ |x - x_0|\ sup_{s \in [x_0 - h\,,x_0 + h]}\ L|y_1(s) - y_2(s)| \\ &\leq Lh\ \|y_1 - y_2\|\ . \end{aligned}$$

Therefore, *T* has a unique fixed point implying that the problem (4.1) has a unique solution.

5 Integral Equations

Now, consider the Fredholm integral equation for an unknown function

$$y: [a\,,b] \to \mathbb{R}\ (-\infty < a < b < \infty)$$

$$y(x) = f(x) + \lambda \int_a^b k(x\,,t)y(t)dt \tag{5.1}$$

where

$$k(x\,,t)\ is\ \ continuous\ \ on\ \ [a\,,b] \times [a\,,b]$$

and

$$f(x)\ is\ \ continuous\ \ on\ \ [a\,,b]$$

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Consider the Banach space $X = C[a\,,b]$ of continuous real-valued functions with supremum norm $\|y\| = \sup\{|y(x)| : x \in [a\,,b]\}$ and define an operator $T: C[a\,,b] \to C[a\,,b]$ by

$$Ty(x) = f(x) + \lambda \int_a^b k(x\,,t)y(t)dt \qquad (5.2)$$

Thus, a solution of Fredholm integral equation (5.1) is a fixed point of *T*.

We now impose a restriction on the real number λ such that *T* becomes a contraction.

Theorem 5.1[1]: Let $k(x\,,t)$ be a continuous function on $[a\,,b] \times [a\,,b]$ with $M = \sup\{|k(x\,,t)| : x\,,t \in [a\,,b]\}$, *f* a contraction function on $[a\,,b]$, and λ a real number such that $M(b-a)|\lambda| < 1$. Then the Fredholm integral equation (5.1) has a unique solution.

Proof: It is sufficient to show that the mapping *T* defined by (5.2) is a contraction. For two continuous functions $y_1\,,y_2 \in C[a\,,b]$, we have

$$\begin{aligned}\|Ty_1 - Ty_2\| &= sup_{x\,\in\,[a\,,b]}|\lambda|\left|\int_a^b k(x\,,t)[y_1(t) - y_2(t)]dt\right| \\ &\le |\lambda| sup_{x\,\in\,[a\,,b]} \int_a^b |k(x\,,t)||y_1(t) - y_2(t)|dt \\ &\le |\lambda|\, M \int_a^b sup_{t\,\in\,[a\,,b]}|y_1(t) - y_2(t)|dt \\ &= |\lambda|\, M\, \|y_1 - y_2\| \int_a^b dt\ .\end{aligned}$$

II. Authors Contribution

We all the three authors have read this article completely and we all have contributed equally towards the completion of this article.

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