Steady State Behavior Of a Network Queue Model Comprised of Two Bi-serial Channels Linked with a Common Server

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***Abstarct*-This paper is an attempt to study the steady –state behavior of a network queuing model in which two bi-serial channels are linked with a common server .The arrivals of service pattern follow Poisson Law .The various queue characteristics such as mean queue length ,variance of the queue length have been obtained explicitly by applying Generating function technique and laws of calculus .The model finds an application in decision making in the process industries ,in banking and in many administrative steps.**

***Keywords*-Poisson Law, Steady State Behavior, Mean Queue Length, Variance and Bi-serial channels.**

I. INTRODUCTION

Jackson [1]studied the behavior of a queuing system containing phase type service. O'Brien[2] studied the transient solution of a queue model comprising of two queues in series in which the service parameter depends upon their queue lengths. Stephan[3] discussed two queues under pre-emptive priority with Poisson arrivals and service rate. Maggu[4] introduced the concept of bitendom in theory of queues which corresponds to a practical situation arises in production concern. Later on this idea was developed by various authors with different modifications in assumptions. Matoori Towfigh and Singh T.P[5] study a network queue model consisting two bi serial channels linked with a common server. Singh T.P. [6] studied the Transient analysis of feedback queue model under service parameter constraint .

Recently, Singh T.P and Kumar Vinod et.al [7] studied the transient behavior of a queuing network with parallel bi serial queues. Further Singh T.P et al [8] studied steady state behavior of a queue model comprised of two subsystems with biserial channels linked with a common channel. Later on Gupta Deepak, Singh T.P et al [9] studied a network queue model comprised of biserial channel linked with a common server. Singh T.P.,Kusum, Deepak Gupta[10] introduced the Feedback Queue Model Assumed Service Rate Proportional to Queue

Number.Gupta,Deepak Gupta[11]studied the concept of steady state solutions of multiple parallel channels in series and non-serial multiple parallel channels both in balking and reneging.

The present queue model differ the study made by Gupta Deepak, Singh T.P. et.al. in the sense that in this model the first and second both the systems consist of biserial channels and both the systems are linked with a common server. The differential difference equations formed in the model have been solved by using generating function technique, laws of calculus and statistical tools in order to find various queue characteristics such as mean queue length, variance of queue etc.

II. PRACTICAL SITUATION

Many practical situations of the model arise in industries, administrative setup , banking system, computer networks , office management , super-market etc. For example: In a mall, there are three sections one is drink section second is for movie and third is food section.Drink section consists of two subsections one is for soft-drink and other is for hot drink. A person may take the soft drink and directly go to watch the movie .Or the person may also take the hot drink and then go to see the movie. After watching the movie the person may go to the food section to eat food. Similarly Food section consists of two subsections one is Indian food sections and other is Chinese food section .After movie,a person may either go to the Indian food section and leave the queue, or the person may go to the Chinese food section.

Further this type of queue network is widely used for modeling and performance prediction in diverse areas as communication networks and teletraffic, multiprogramming systems, maintenance and repair facilities, production and assembly lines, steel making , air traffic control , urban transportation systems etc.

III. THE PROBLEM

The entire queue model is comprised of three service channels S_1, S_2 and S_3.

Where subsystem S_1 consists of two biserial service channels s_{11} and s_{12} and also S_3 contains two biserial channels s_{21} and s_{22}. S_1 and S_3 are linked with a common server S_2 in between both. The service time at S_{ij} are distributed exponentially. For convenience, we assume the service rate $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$.for μ_{ij} at S_{ij} respectively. The customers initially join S_{ij} under poisson assumptions and the mean rate λ_1, λ_2 respectively. Queues are formed in the front of the service channels S_{11} and S_{12} if they are busy. Customers coming at the rate λ_1 after completion of phase service at S_{11} will join S_{12} or S_2 i.e either they may go to the network of server S_{12} with the probability p_{12} or go to the server S_2 with the probabillty p_{13} such that $p_{12} + p_{13}$=1.And the customers coming at the rate λ_1 after completion of phase service at S_{12} will join S_{11} or S_2 i.e either they may go to the network of server S_{11} with the probability p_{21} or go to the server S_2 with the probabillty p_{23} such that $p_{21} + p_{23}$=1.After that customer will go from the server S_2 either to the server S_{21} with the probability p_{34} or go to the server S_{22} with the probability p_{35} such that $p_{34} + p_{35} = 1$

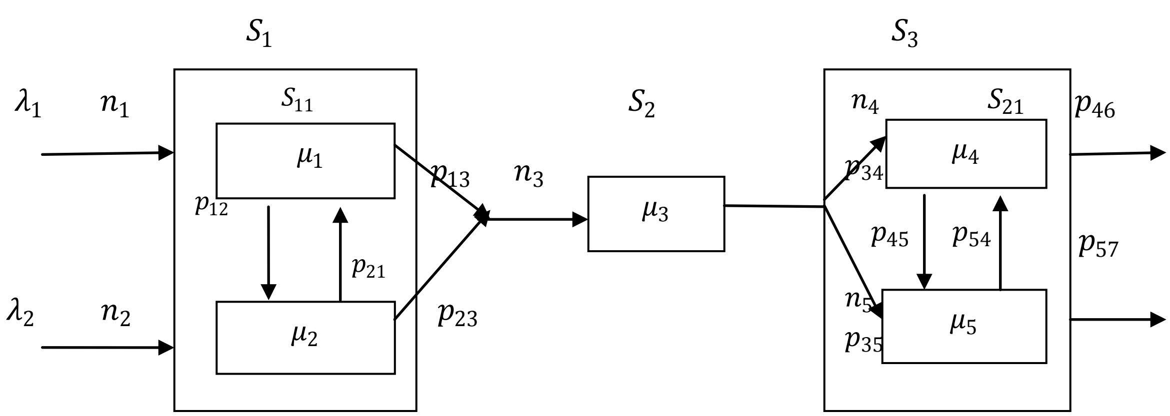

Fig :1

IV. MATHEMATICAL ANALYSIS

Define P_{n_1,n_2,n_3,n_4,n_5} the probability ,there are n_1 units wating in the queue Q_1 in front of S_{11} , n_2 units waiting in the queue Q_2 in front of S_{12} , n_3 units waiting in the queue Q_3 in front of S_2,and n_4 units wating in the queue Q_4 in front of S_{21} , n_5 units waiting in the queue Q_5 in front of S_{22}.In each case the waiting includes a unit in service ,if any $(n_1, n_2, n_3, n_4, n_5 > 0)$

Differential Difference Equation in Transient form for the model is:

$P_{n_1,n_2,n_3,n_4,n_5} = -(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)\,P_{n_1,n_2,n_3,n_4,n_5}(t)+\lambda_1\,P_{n_1-1,n_2,n_3,n_4,n_5}(t)$
$+\lambda_2\,P_{n_1,n_2-1,n_3,n_4,n_5}(t)+\mu_1 p_{12}(n_1+1)\,P_{n_1+1,n_2-1,n_3,n_4,n_5}(t)+\mu_1 p_{13}(n_1+1)\,P_{n_1+1,n_2,n_3-1,n_4,n_5}(t)+\mu_2 p_{21}(n_2+1)\,P_{n_1-1,n_2+1,n_3,n_4,n_5}(t)+\mu_2 p_{23}(n_2+1)\,P_{n_1,n_2+1,n_3-1,n_4,n_5}(t)+\mu_3 p_{34}(n_3+1)\,P_{n_1,n_2,n_3+1,n_4-1,n_5}(t)+\mu_3 p_{35}(n_3+1)\,P_{n_1,n_2,n_3+1,n_4,n_5-1}(t)+\mu_4 p_{45}(n_4+1)\,P_{n_1,n_2,n_3,n_4+1,n_5-1}(t)+\mu_4 p_{46}(n_4+1)\,P_{n_1,n_2,n_3,n_4+1,n_5}(t)+\mu_5 p_{54}(n_5+1)\,P_{n_1,n_2,n_3,n_4-1,n_5+1}(t)+\mu_5 p_{57}(n_5+1)\,P_{n_1,n_2,n_3,n_4,n_5+1}(t)$

Equation in the steady state form:

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)\,P_{n_1,n_2,n_3,n_4,n_5}$

$=\lambda_1\,P_{n_1-1,n_2,n_3,n_4,n_5}+\lambda_2\,P_{n_1,n_2-1,n_3,n_4,n_5}+\mu_1 p_{12}\,P_{n_1+1,n_2-1,n_3,n_4,n_5}+\mu_1 p_{13}\,P_{n_1+1,n_2,n_3-1,n_4,n_5}+\mu_2 p_{21}\,P_{n_1-1,n_2+1,n_3,n_4,n_5}+\mu_2 p_{23}\,P_{n_1,n_2+1,n_3-1,n_4,n_5}+\mu_3 p_{34}\,P_{n_1,n_2,n_3+1,n_4-1,n_5}+\mu_3 p_{35}\,P_{n_1,n_2,n_3+1,n_4,n_5-1}+\mu_4 p_{45}P_{n_1,n_2,n_3,n_4+1,n_5-1}+\mu_4 p_{46}\,P_{n_1,n_2,n_3,n_4+1,n_5}+\mu_5 p_{54}\,P_{n_1,n_2,n_3,n_4-1,n_5+1}+\mu_5 p_{57}\,P_{n_1,n_2,n_3,n_4,n_5+1}$

$(n_1, n_2, n_3, n_4, n_5 > 0)$ (1)

$(\lambda_1+\lambda_2+\mu_2+\mu_3+\mu_4+\mu_5)\,P_{0,n_2,n_3,n_4,n_5}$

$=\lambda_2\,P_{0,n_2-1,n_3,n_4,n_5}+\mu_1 p_{12}\,P_{1,n_2-1,n_3,n_4,n_5}+\mu_1 p_{13}\,P_{1,n_2,n_3-1,n_4,n_5}+\mu_2 p_{23}\,P_{0,n_2+1,n_3-1,n_4,n_5}+\mu_3 p_{34}\,P_{0,n_2,n_3+1,n_4-1,n_5}+\mu_3 p_{35}\,P_{0,n_2,n_3+1,n_4,n_5-1}+\mu_4 p_{45}P_{0,n_2,n_3,n_4+1,n_5-1}+\mu_4 p_{46}\,P_{0,n_2,n_3,n_4+1,n_5}+\mu_5 p_{54}\,P_{0,n_2,n_3,n_4-1,n_5+1}+p_{57}\,P_{0,n_2,n_3,n_4,n_5+1}$

$(n_1 = 0, n_2, n_3, n_4, n_5 > 0)$ (2)

$(\lambda_1+\lambda_2+\mu_3+\mu_4+\mu_5)\,P_{n_1,0,n_3,n_4,n_5}$

$=\lambda_1\,P_{n_1-1,0,n_3,n_4,n_5}+\mu_1 p_{13}\,P_{n_1+1,0,n_3-1,n_4,n_5}+\mu_2 p_{21}\,P_{n_1-1,1,n_3,n_4,n_5}+\mu_2 p_{23}\,P_{n_1,1,n_3-1,n_4,n_5}+\mu_3 p_{34}\,P_{n_1,0,n_3+1,n_4-1,n_5}+\mu_3 p_{35}\,P_{n_1,0,n_3+1,n_4,n_5-1}+\mu_4 p_{45}P_{n_1,0,n_3,n_4+1,n_5-1}+\mu_4 p_{46}\,P_{n_1,0,n_3,n_4+1,n_5}+\mu_5 p_{54}\,P_{n_1,0,n_3,n_4-1,n_5+1}+\mu_5 p_{57}\,P_{n_1,0,n_3,n_4,n_5+1}$

$(n_2 = 0, n_1, n_3, n_4, n_5 > 0)$ (3)

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_4+\mu_5)\,P_{n_1,n_2,0,n_4,n_5}$

$=\lambda_1\,P_{n_1-1,n_2,0,n_4,n_5}+\lambda_2\,P_{n_1,n_2-1,0,n_4,n_5}+\mu_1 p_{12}\,P_{n_1+1,n_2-1,0,n_4,n_5}+\mu_2 p_{21}\,P_{n_1-1,n_2+1,0,n_4,n_5}+\mu_3 p_{34}\,P_{n_1,n_2,1,n_4-1,n_5}+\mu_3 p_{35}\,P_{n_1,n_2,1,n_4,n_5-1}+\mu_4 p_{45}P_{n_1,n_2,0,n_4+1,n_5-1}+\mu_4 p_{46}\,P_{n_1,n_2,0,n_4+1,n_5}+\mu_5 p_{54}\,P_{n_1,n_2,0,n_4-1,n_5+1}+\mu_5 p_{57}\,P_{n_1,n_2,0,n_4,n_5+1}$

$(n_3 = 0, n_1, n_2, n_4, n_5 > 0)$ (4)

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_5)\,P_{n_1,n_2,n_3,0,n_5}$

$=\lambda_1\,P_{n_1-1,n_2,n_3 0,n_5}+\lambda_2\,P_{n_1,n_2-1,0,n_5}+\mu_1 p_{12}\,P_{n_1+1,n_2-1,n_3 0,n_5}+\mu_1 p_{13}\,P_{n_1+1,n_2,n_3-1,0,n_5}+\mu_2 p_{21}\,P_{n_1-1,n_2+1,n_3,0,n_5}+\mu_2 p_{23}\,P_{n_1,n_2+1,n_3-1,0,n_5}+\mu_3 p_{35}\,P_{n_1,n_2,n_3+1,0,n_5-1}+\mu_4 p_{45}P_{n_1,n_2,n_3,1,n_5-1}+\mu_4 p_{46}\,P_{n_1,n_2,n_3,1,n_5}+\mu_5 p_{57}\,P_{n_1,n_2,n_3,0,n_5+1}$

$(n_4 = 0, n_1, n_2, n_3, n_5 > 0)$ (5)

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4)\,P_{n_1,n_2,n_3,n_4,0}$

$=\lambda_1 P_{n_1-1,n_2,n_3,n_4,0}+\lambda_2 P_{n_1,n_2-1,n_3,n_4,0}+\mu_1 p_{12} P_{n_1+1,n_2-1,n_3,n_4,0}+\mu_1 p_{13} P_{n_1+1,n_2,n_3-1,n_4,0}+\mu_2 p_{21} P_{n_1-1,n_2+1,n_3,n_4,0}+\mu_2 p_{23} P_{n_1,n_2+1,n_3-1,n_4,0}+$
$\mu_3 p_{34} P_{n_1,n_2,n_3+1,n_4-1,0}+\mu_4 p_{46} P_{n_1,n_2,n_3,n_4+1,0}+\mu_5 p_{54} P_{n_1,n_2,n_3,n_4-1,1}+\mu_5 p_{57} P_{n_1,n_2,n_3,n_4,1}$

$(n_5 = 0, n_1, n_2, n_3, n_4, > 0)$ (6)

$(\lambda_1+\lambda_2+\mu_3+\mu_4+\mu_5) P_{0,0,n_3,n_4,n_5}$

$=\mu_1 p_{13} P_{1,0,n_3-1,n_4,n_5}+\mu_2 p_{23} P_{0,1,n_3-1,n_4,n_5}+\mu_3 p_{34} P_{0,0,n_3+1,n_4-1,n_5}+\mu_3 p_{35} P_{0,0,n_3+1,n_4,n_5-1}+\mu_4 p_{45} P_{0,0,n_3,n_4+1,n_5-1}+\mu_4 p_{46} P_{0,0,n_3,n_4+1,n_5}+$
$\mu_5 p_{54} P_{0,0,n_3,n_4-1,n_5+1}+\mu_5 p_{57} P_{0,0,n_3,n_4,n_5+1}$

$(n_1, n_2 = 0, n_3, n_4, n_5 > 0)$ (7)

$(\lambda_1+\lambda_2+\mu_2+\mu_4+\mu_5) P_{0,n_2,0,n_4,n_5}$
$=\lambda_2 P_{0,n_2-1,0,n_4,n_5}+\mu_1 p_{12} P_{1,n_2-1,0,n_4,n_5}+\mu_2 p_{21} P_{n_1-1,n_2+1,n_3,n_4,n_5}+\mu_3 p_{34} P_{0,n_2,1,n_4-1,n_5}+\mu_3 p_{35} P_{0,n_2,1,n_4,n_5-1}+\mu_4 p_{45} P_{0,n_2,0,n_4+1,n_5-1}+$
$\mu_4 p_{46} P_{0,n_2,0,n_4+1,n_5}+\mu_5 p_{54} P_{0,n_2,0,n_4-1,n_5+1}+\mu_5 p_{57} P_{0,n_2,0,n_4,n_5+1}$

$(n_1, n_3 = 0, n_2, n_4, n_5 > 0)$ (8)

$(\lambda_1+\lambda_2+\mu_2+\mu_3+\mu_5) P_{0,n_2,n_3,0,n_5}$
$=\lambda_2 P_{,n_2-1,n_3,0,n_5}+\mu_1 p_{12} P_{1,n_2-1,n_3,0,n_5}+\mu_1 p_{13} P_{1,n_2,n_3-1,0,n_5}+\mu_2 p_{23} P_{n_1,n_2+1,n_3-1,n_4,n_5}+\mu_3 p_{35} P_{0,n_2,n_3+1,0,n_5-1}+\mu_4 p_{45} P_{0,n_2,n_3,1,n_5-1}+$
$\mu_4 p_{46} P_{0,n_2,n_3,1,n_5}+\mu_5 p_{57} P_{0,n_2,n_3,0,n_5+1}$

$(n_1, n_4 = 0, n_2, n_3, n_5 > 0)$ (9)

$(\lambda_1+\lambda_2+\mu_2+\mu_3+\mu_4) P_{0,n_2,n_3,n_4,0}$
$=\lambda_2 P_{0,n_2-1,n_3,n_4,0}+\mu_1 p_{12} P_{1,n_2-1,n_3,n_4,0}+\mu_1 p_{13} P_{1,n_2,n_3-1,n_4,0}+\mu_2 P_{23} P_{0,n_2+1,n_3-1,n_4,0}+\mu_3 P_{34} P_{0,n_2,n_3+1,n_4-1,0}+\mu_4 P_{46} P_{0,n_2,n_3,n_4+1,0}+$
$\mu_5 P_{54} P_{0,n_2,n_3,n_4-1,1}+\mu_5 P_{57} P_{0,n_2,n_3,n_4,1}$

$(n_1, n_5 = 0, n_2, n_3, n_4 > 0)$ (10)

$(\lambda_1+\lambda_2+\mu_1+\mu_4+\mu_5) P_{n_1,0,0,n_4,n_5}$

$=\lambda_1 P_{n_1-1,0,0,n_4,n_5}+\mu_2 p_{21} P_{n_1-1,1,0,n_4,n_5}+\mu_3 p_{34} P_{n_1,0,1,n_4-1,n_5}+\mu_3 p_{35} P_{n_1,0,1,n_4,n_5-1}+\mu_4 p_{45} P_{n_1,0,0,n_4+1,n_5-1}+\mu_4 p_{46} P_{n_1,0,0,n_4+1,n_5}+$
$\mu_5 p_{54} P_{n_1,0,0,n_4-1,n_5+1}+\mu_5 p_{57} P_{n_1,0,0,n_4,n_5+1}$

$(n_2, n_3 = 0, n_1, n_4, n_5 > 0)$ (11)

$(\lambda_1+\lambda_2+\mu_1+\mu_3+\mu_5) P_{n_1,0,n_3,0,n_5}$

$=\lambda_1 P_{n_1-1,0,n_3,0,n_5}+\mu_1 p_{13} P_{n_1+1,0,n_3-1,0,n_5}+\mu_2 p_{21} P_{n_1-1,1,n_3,0,n_5}+\mu_2 p_{23} P_{n_1,1,n_3-1,0,n_5}+\mu_3 p_{35} P_{n_1,0,n_3+1,0,n_5-1}+\mu_4 p_{45} P_{n_1,0,n_3,1,n_5-1}+$
$\mu_4 p_{46} P_{n_1,0,n_3,1,n_5}+\mu_5 p_{54} P_{n_1,n_2,n_3,n_4-1,n_5+1}+\mu_5 p_{57} P_{n_1,0,n_3,0,n_5+1}$

$(n_2, n_4 = 0, n_1, n_3, n_5 > 0)$ (12)

$(\lambda_1+\lambda_2+\mu_1+\mu_3+\mu_4) P_{n_1,0,n_3,n_4,0}$

$=\lambda_1 P_{n_1-1,0,n_3,n_4,0}+\mu_1 p_{13} P_{n_1+1,0,n_3-1,n_4,0}+\mu_2 p_{21} P_{n_1-1,1,n_3,n_4,0}+\mu_2 p_{23} P_{n_1,1,n_3-1,n_4,0}+\mu_3 p_{34} P_{n_1,0,n_3+1,n_4-1,0}+\mu_4 p_{46} P_{n_1,0,n_3,n_4+1,0}+$
$\mu_5 p_{54} P_{n_1,0,n_3,n_4-1,1}+\mu_5 p_{57} P_{n_1,0,n_3,n_4,1}$

$(n_2\ , n_5{=}0, n_1, n_3, n_4 > 0)$ (13)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_5)\, P_{n_1,n_2,0,0,n_5}$

$=\lambda_1\ P_{n_1-1,n_2,0,0,n_5}+\lambda_2\ P_{n_1,n_2-1,0,0,n_5}+\mu_1 p_{12}\ P_{n_1+1,n_2-1,0,0,n_5}+\mu_2 p_{21}\ P_{n_1-1,n_2+1,0,0,n_5}+\mu_3 p_{35}\ P_{n_1,n_2,1,0,n_5-1}+\mu_4 p_{45} P_{n_1,n_2,0,1,n_5-1}+\mu_4 p_{46}\ P_{n_1,n_2,0,1,n_5}+$
$\mu_5 p_{57}\ P_{n_1,n_2,0,0,n_5+1}$

$(n_3\ , n_4{=}0, n_1, n_2, n_5 > 0)$ (14)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_4)\, P_{n_1,n_2,0,n_4,0}$

$=\lambda_1\ P_{n_1-1,n_2,0,n_4,0}+\lambda_2\ P_{n_1,n_2-1,0,n_4,0}+\mu_1 p_{12}\ P_{n_1+1,n_2-1,0,n_4,0}+\mu_2 p_{21}\ P_{n_1-1,n_2+1,0,n_4,0}+\mu_3 p_{34}\ P_{n_1,n_2,1,n_4-1,0}+\mu_4 p_{46}\ P_{n_1,n_2,0,n_4+1,0}+\mu_5 p_{54}\ P_{n_1,n_2,0,n_4-1,1}$
$+\mu_5 p_{57}\ P_{n_1,n_2,0,n_4,1}$

$(n_3\ , n_5{=}0, n_1, n_2, n_4 > 0)$ (15)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3)\, P_{n_1,n_2,n_3,0,0}$

$=\lambda_1\ P_{n_1-1,n_2,n_3,0,0}+\lambda_2\ P_{n_1,n_2-1,n_3,0,0}+\mu_1 p_{12}\ P_{n_1+1,n_2-1,n_3,0,0}+\mu_1 p_{13}\ P_{n_1+1,n_2,n_3-1 0,0}+\mu_2 p_{21}\ P_{n_1-1,n_2+1,n_3,0,0}+\mu_2 p_{23}\ P_{n_1,n_2+1,n_3-1,0,0}+$
$\mu_4 p_{46}\ P_{n_1,n_2,n_3,1,0}+\mu_5 p_{57}\ P_{n_1,n_2,n_3,0,1}$

$(n_4\ , n_5{=}0, n_1, n_2, n_3 > 0)$ (16)

$(\lambda_1 + \lambda_2 + \mu_4 + \mu_5)\, P_{0,0,0,n_4,n_5}$

$=\mu_3 p_{34}\ P_{0,0,1,n_4-1,n_5}+\mu_3 p_{35}\ P_{0,0,1,n_4,n_5-1}+\mu_4 p_{45} P_{0,0,0,n_4+1,n_5-1}+\mu_4 p_{46}\ P_{0,0,0,n_4+1,n_5}+\mu_5 p_{54}\ P_{0,0,0,n_4-1,n_5+1}+\mu_5 p_{57}\ P_{0,0,0,n_4,n_5+1}$

$(n_1\ , n_2, n_3 = 0 n_4, n_5 > 0)$ (17)

$(\lambda_1 + \lambda_2 + \mu_3 + \mu_5)\, P_{0,0,n_3,0,n_5}$

$=\mu_1 p_{13}\ P_{1,0,n_3-1,0,n_5}+\mu_2 p_{23}\ P_{0,1,n_3-1,0,n_5}+\mu_3 p_{35}\ P_{0,0,n_3+1,0,n_5-1}+\mu_4 p_{45} P_{0,0,n_3,1,n_5-1}+\mu_4 p_{46}\ P_{0,0,n_3,1,n_5}+\mu_5 p_{57}\ P_{0,0,n_3,0,n_5+1}$

$(n_1\ , n_2, n_4 = 0 n_3, n_5 > 0)$ (18)

$(\lambda_1 + \lambda_2 + \mu_3 + \mu_4)\, P_{0,0,n_3,n_4,0}$

$=\mu_1 p_{13}\ P_{1,0,n_3-1,n_4,0}+\mu_2 p_{23}\ P_{0,1,n_3-1,n_4,0}+\mu_3 p_{34}\ P_{0,0,n_3+1,n_4-1,0}+\mu_4 p_{46}\ P_{0,0,n_3,n_4+1,0}+\mu_5 p_{54}\ P_{0,0,n_3,n_4-1,1}+\mu_5 p_{57}\ P_{0,0,n_3,n_4,1}$

$(n_1\ , n_2, n_5 = 0 n_3, n_4 > 0)$ (19)

$(\lambda_1 + \lambda_2 + \mu_2 + \mu_5)\, P_{0,n_2,0,0,n_5}$

$=\lambda_2\ P_{0,n_2-1,00,n_5}+\mu_1 p_{12}\ P_{1,n_2-1,0,0,n_5}+\mu_3 p_{35}\ P_{0,n_2,1,0,n_5-1}+\mu_4 p_{45} P_{0,n_2,0,1,n_5-1}+\mu_4 p_{46}\ P_{0,n_2,0,1,n_5}+\mu_5 p_{57}\ P_{0,n_2,0,0,n_5+1}$
$(n_1\ , n_3, n_4 = 0 n_2, n_5 > 0)$ (20)

$(\lambda_1 + \lambda_2 + \mu_2 + \mu_4)\, P_{0,n_2,0,n_4,0}$

$=\lambda_2\ P_{0,n_2-1,0n_4,0}+\mu_1 p_{12}\ P_{1,n_2-1,0,n_4,0}+\mu_3 p_{34}\ P_{0,n_2,1,n_4-1,0}+\mu_4 p_{46}\ P_{0,n_2 0,n_4+1,0}+\mu_5 p_{54}\ P_{0,n_2,0,n_4-1,1}+\mu_5 p_{57}\ P_{0,n_2,0,n_4,1}$

$(n_1\ , n_3, n_5 = 0 n_2, n_4 > 0)$ (21)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5)\, P_{0,n_2,n_3,0,0}$

$=\lambda_2\, P_{1,n_2-1,n_3,0,0} + \mu_1 p_{12}\, P_{1,n_2-1,n_3,0,0} + \mu_1 p_{13}\, P_{1,n_2,n_3-1,0,0} + \mu_2 p_{23}\, P_{0,n_2+1,n_3-1,0,0} + \mu_4 p_{46}\, P_{n_1,n_2,n_3,1,0} + \mu_5 p_{57}\, P_{0,n_2,n_3,0,1}$

$(n_1, n_4, n_5 = 0 n_2, n_3 > 0)$ (22)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_5)\, P_{n_1,0,0,0,n_5}$

$=\lambda_1\, P_{n_1-1,0,0,0,n_5} + \mu_2 p_{21}\, P_{n_1-1,1,0,0,n_5} + \mu_3 p_{35}\, P_{n_1,0,1,0,n_5-1} + \mu_4 p_{45} P_{n_1,0,0,1,n_5-1} + \mu_4 p_{46}\, P_{n_1,0,0,1,n_5} + \mu_5 p_{57}\, P_{n_1,0,0,0,n_5+1}$

$(n_2, n_3, n_4 = 0 n_1, n_5 > 0)$ (23)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_4)\, P_{n_1,0,0,n_4,0}$

$=\lambda_1\, P_{n_1-1,0,0,n_4,0} + \mu_2 p_{21}\, P_{n_1-1,1,0,n_4,0} + \mu_3 p_{34}\, P_{n_1,0,1,n_4-1,0} + \mu_4 p_{46}\, P_{n_1,0,0,n_4+1,0} + \mu_5 p_{54}\, P_{n_1,0,0,n_4-1,1} + \mu_5 p_{57}\, P_{n_1,0,0,n_4,1}$

$(n_2, n_3, n_5 = 0 n_1, n_4 > 0)$ (24)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_3)\, P_{n_1,0,n_3,0,0}$

$=\lambda_1\, P_{n_1-1,0,n_3,0,0} + \mu_1 p_{13}\, P_{n_1+1,0,n_3-1,0,0} + \mu_2 p_{21}\, P_{n_1-1,1,n_3,0,0} + \mu_2 p_{23}\, P_{n_1,1,n_3-1,0,0} + \mu_4 p_{46}\, P_{n_1,0,n_3,1,0} + \mu_5 p_{57}\, P_{n_1,0,n_3,0,1}$

$(n_2, n_4, n_5 = 0 n_1, n_3 > 0)$ (25)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\, P_{n_1,n_2,0,0,0}$

$=\lambda_1\, P_{n_1-1,n_2,0,0,0} + \lambda_2\, P_{n_1,n_2-1,0,0,0} + \mu_1 p_{12}\, P_{n_1+1,n_2-1,0,0,0} + \mu_2 p_{21}\, P_{n_1-1,n_2+1,0,0,0} + \mu_4 p_{46}\, P_{n_1,n_2,0,1,0} + \mu_5 p_{57}\, P_{n_1,n_2,0,01}$

$(n_3, n_4, n_5 = 0 n_1, n_2 > 0)$ (26)

$(\lambda_1 + \lambda_2 + \mu_5)\, P_{0,0,0,0,n_5}$

$=\mu_3 p_{35}\, P_{0,0,1,0,n_5-1} + \mu_4 p_{45} P_{0,0,0,1,n_5-1} + \mu_4 p_{46}\, P_{0,0,0,1,n_5} + \mu_5 p_{57}\, P_{0,0,0,0,n_5+1}$

$(n_1, n_2, n_3, n_4 = 0, n_5 > 0)$ (27)

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5)\, P_{0,0,0,n_4,0}$

$=\mu_3 p_{34}\, P_{0,0,1,n_4-1,0} + \mu_4 p_{46}\, P_{0,0,0,n_4+1,0} + \mu_5 p_{54}\, P_{0,0,0,n_4-1,1} + \mu_5 p_{57}\, P_{0,0,0,n_4,1}$

$(n_1, n_2, n_3, n_5 = 0, n_4 > 0)$ (28)

$(\lambda_1 + \lambda_2 + \mu_3)\, P_{0,0,n_3,0,0}$

$=\mu_1 p_{13}\, P_{1,0,n_3-1,0,0} + \mu_2 p_{23}\, P_{0,1,n_3-1,0,0} + \mu_4 p_{46}\, P_{0,0,n_3,1,0} + \mu_5 p_{57}\, P_{0,0,n_3,0,1}$

$(n_1, n_2, n_4, n_5 = 0, n_3 > 0)$ (29)

$(\lambda_1 + \lambda_2 + \mu_2)\, P_{0,n_2,0,0,0}$

$=\lambda_2\, P_{0,n_2-1,0,0,0} + \mu_1 p_{12}\, P_{1,n_2-1,0,0,0} + \mu_4 p_{46}\, P_{0,n_2,0,1,0} + \mu_5 p_{57}\, P_{0,n_2,0,0,1}$

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$(n_1, n_3, n_4, n_5 = 0, n_2 > 0)$ (30)

$(\lambda_1 + \lambda_2 + \mu_1)\, P_{n_1,0,0,0,0}$

$=\lambda_1\, P_{n_1-1,0,0,0,0} + \mu_2 p_{21}\, P_{n_1-1,1,0,0,0} + \mu_4 p_{46}\, P_{n_1,0,0,1,0} + \mu_5 p_{57}\, P_{n_1,0,0,0,1}$

$(n_1, n_3, n_4, n_5 = 0, n_2 > 0)$ (31)

$(\lambda_1 + \lambda_2)\, P_{0,0,0,0,0} = \mu_4 p_{46}\, P_{0,0,0,1,0} + \mu_5 p_{57}\, P_{0,0,0,0,1}$

$(n_1, n_2, n_3, n_4, n_5 = 0)$ (32)

Now Define Generating Function as

$F(X,Y,Z,R,S) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5}\, X^{n_1} Y^{n_2} Z^{n_3} R^{n_4} S^{n_5}$

(33) Where $|X|, |Y|, |Z|, |R|, |S| = 1$

Also define partial generating function as

$F_{n_2,n_3,n_4,n_5}(X) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5}\, X^{n_1}$ (34)

$F_{n_3,n_4,n_5}(X,Y) = \sum_{n_2=0}^{\infty} F_{n_2,n_3,n_4,n_5}(X) Y^{n_2}$ (35)

$F_{n_4,n_5}(X,Y,Z) = \sum_{n_3=0}^{\infty} F_{n_3,n_4,n_5}(X,Y) Z^{n_3}$ (36)

$F_{n_5}(X,Y,Z,R) = \sum_{n_4=0}^{\infty} F_{n_4,n_5}(X,Y,Z) R^{n_4}$ (37)

$F(X,Y,Z,R,S) = \sum_{n_5=0}^{\infty} F_{n_5}(X,Y,Z,R) S^{n_5}$ (38)

Now proceeding on the lines of Maggu, Singh T.P. et.al. and following the standard technique ,which after manipulation gives the final reduced results as-

$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5)F(X,Y,Z,R,S) - \mu_5 F(X,Y,Z,R) - \mu_4 F(X,Y,Z,S) - \mu_3 F(X,Y,R,S) - \mu_2 F(X,Z,R,S) - \mu_1 F(Y,Z,R,S) = \lambda_1 X f(X,Y,Z,R,S) + \lambda_2 X f(X,Y,Z,R,S) + \frac{\mu_1 p_{12} Y}{X}[F(X,Y,Z,R,S) - F(Y,Z,R,S)] +$

$\frac{\mu_1 p_{13} Z}{X}[F(X,Y,Z,R,S) - F(Y,Z,R,S)] + \frac{\mu_2 p_{21} X}{Y}[F(X,Y,Z,R,S) - F(X,Z,R,S)] +$

$\frac{\mu_2 p_{23} Z}{Y}[F(X,Y,Z,R,S) - F(X,Z,R,S)] + \frac{\mu_3 p_{34} R}{Z}[F(X,Y,Z,R,S) - F(X,Y,R,S)] + \frac{\mu_3 p_{35} S}{Z}[F(X,Y,Z,R,S) - F(X,Y,R,S)]$
$+ \frac{\mu_4 p_{45} S}{R}[F(X,Y,Z,R,S) - F(X,Y,Z,S)] +$

$\frac{\mu_4 p_{46}}{R}[F(X,Y,Z,R,S) - F(X,Y,Z,S)] + \frac{\mu_5 p_{54} R}{S}[F(X,Y,Z,R,S) - F(X,Y,Z,S)] +$

$\frac{\mu_5 p_{57}}{S}[F(X,Y,Z,R,S) - F(X,Y,Z,S)]$ (39)

$(\lambda_1(1-X) + \lambda_2(1-Y) + \mu_1\left(1 - p_{12}\frac{Y}{X} - p_{13}\frac{Z}{X}\right) + \mu_2\left(1 - p_{21}\frac{X}{Y} - p_{23}\frac{Z}{Y}\right) + \mu_3(1 - p_{34}\frac{R}{Z} - p_{35}\frac{S}{Z}) + \mu_4(1 - p_{45}\frac{S}{R} - \frac{p_{46}}{R}) + \mu_5(1 - p_{54}\frac{R}{S} - \frac{p_{57}}{S}))F(X,Y,Z,R,S) = \mu_5 F(X,Y,Z,R)\,(1 - p_{54}\frac{R}{S} - \frac{p_{57}}{S}) + \mu_4 F(X,Y,Z,S)\,(1 - p_{45}\frac{S}{R} - \frac{p_{46}}{R}) + \mu_3 F(X,Y,R,S)\,(1 - p_{34}\frac{R}{Z} - p_{35}\frac{S}{Z}) + \mu_2 F(X,Z,R,S)\left(1 - p21XY - p23ZY + \mu 1 F(Y,Z,R,S)\ 1 - p12YX - p13ZX\right.$ (40)

F(X,Y,Z,R,S)=

$$\frac{\left[\begin{array}{c}\mu_5 F(X,Y,Z,R)\left(1-p_{54}\frac{R}{S}-\frac{p_{57}}{S}\right)+\mu_4 F(X,Y,Z,S)\left(1-p_{45}\frac{S}{R}-\frac{p_{46}}{R}\right)+\mu_3 F(X,Y,R,S)\left(1-p_{34}\frac{R}{Z}-p_{35}\frac{S}{Z}\right)\\ +\mu_2 F(X,Z,R,S)\left(1-p_{21}\frac{X}{Y}-p_{23}\frac{Z}{Y}\right)+\mu_1 F(Y,Z,R,S)\left(1-p_{12}\frac{Y}{X}-p_{13}\frac{Z}{X}\right)\end{array}\right]}{\begin{array}{c}(\lambda_1(1-X)+\lambda_2(1-Y)+\mu_1\left(1-p_{12}\frac{Y}{X}-p_{13}\frac{Z}{X}\right)+\mu_2\left(1-p_{21}\frac{X}{Y}-p_{23}\frac{Z}{Y}\right)\\ +\mu_3(1-p_{34}\frac{R}{Z}-p_{35}\frac{S}{Z})+\mu_4(1-p_{45}\frac{S}{R}-\frac{p_{46}}{R})+\mu_5(1-p_{54}\frac{R}{S}-\frac{p_{57}}{S}))\end{array}} \quad (41)$$

Let us denote :- $F(Y,Z,R,S)=F_1$

$F(X,Z,R,S)=F_2$

$F(X,Y,R,S)=F_3$

$F(X,Y,Z,S)=F_4$

$F(X,Y,Z,R)=F_5$

Also $F(1,1,1,1,1)=1$ Total probability ,

Let (X=1) as Y,Z,R,S→1

Now (41) becomes $\frac{0}{0}$ form .Now by using L'Hospital rule ,we get by differentiating w.r.t.X

$$1=\frac{\mu_1 F_1(p_{13}+p_{12})+\mu_2 F_2(-p_{21})}{-\lambda_1+\mu_1(p_{12}+p_{13})+\mu_2(-p_{21})}$$

$$-\lambda_1+\mu_1-\mu_2 p_{21}=\mu_1 F_1-\mu_2 F_2 p_{21}$$

$$\text{where } p_{13}+p_{12}=1 \quad (42)$$

Again differentiating w.r.t.Y gives Y=1

$$1=\frac{\mu_2 F_2(p_{23}+p_{21})+\mu_1 F_1(-p_{12})}{-\lambda_2+\mu_2(p_{21}+p_{23})+\mu_1(-p_{12})}$$

$$\Rightarrow \quad -\lambda_2-\mu_1 p_{12}+\mu_2=-\mu_1 F_1 p_{12}+\mu_2 F_2 \quad (43)$$

where $p_{21}+p_{23}=1$

Again differentiating w.r.t.Z gives Z=1

$$1=\frac{\mu_3 F_3(p_{34}+p_{35})+\mu_2 F_2(-p_{23})+\mu_1 F_1(-p_{13})}{\mu_3(p_{34}+p_{35})+\mu_2(-p_{23})+\mu_1(-p_{13})}$$

$$\Rightarrow \quad -\mu_1 p_{13}-\mu_2 p_{23}+\mu_3=-\mu_1 p_{13} F_1-\mu_2 p_{23} F_2+\mu_3 F_3 \quad (44)$$

where $p_{34}+p_{35}=1$

Again differentiating w.r.t.R gives R=1

$$1=\frac{\mu_4 F_4(p_{45}+p_{46})+\mu_3 F_3(-p_{34})+\mu_5 F_5(-p_{54})}{\mu_4(p_{45}+p_{46})+\mu_3(-p_{34})+\mu_5(-p_{54})}$$

$$\Rightarrow \quad -\mu_3 p_{34}-\mu_5 p_{54}+\mu_4=-\mu_3 p_{34} F_3-\mu_5 p_{54} F_5+\mu_4 F_4 \quad (45)$$

Where $p_{45}+p_{46}=1$

Again differentiating w.r.t.S gives S=1

$1=\frac{\mu_5 F_5(p_{54}+p_{57})+\mu_3 F_3(-p_{35})+\mu_4 F_4(-p_{45})}{\mu_5(p_{54}+p_{57})+\mu_3(-p_{35})+\mu_4(-p_{45})}$

$$\mu_5 F_5-\mu_4 F_4(p_{45})-\mu_3 F_3(p_{35})=-\mu_3(p_{35})-\mu_4(p_{45})+\mu_5 \quad (46)$$

After solving the above for F_1, F_2, F_3, F_4, F_5 , we get

$F_1=1-\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_1}$

$F_2=1-\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_2}$

$F_3=1-\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3}$

$F_4=1-\left[(p_{34}+p_{35}p_{54})\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_4(1-p_{45}p_{54})}\right]\right]$

$F_5=1-\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})}\right]\left[\frac{p_{45}(p_{34}+p_{35}p_{54})+p_{35}}{\mu_5(1-p_{45}p_{54})}\right]$

V. SOLUTION OF MODEL

$P_{n_1,n_2,n_3,n_4,n_5}=\rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}\rho_4^{n_4}\rho_5^{n_5}\ (1-\rho_1)\ (1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)$

Where

$\rho_1=\frac{\lambda_1+\lambda_2 p_{21}}{(1-p_{21}p_{12})\mu_1}$

$\rho_2=\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_2}$

$\rho_3=\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3}$

$\rho_4=\left[(p_{34}+p_{35}p_{54})\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_4(1-p_{45}p_{54})}\right]\right]$

$\rho_5=\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})}\right]\left[\frac{p_{45}(p_{34}+p_{35}p_{54})+p_{35}}{\mu_5(1-p_{45}p_{54})}\right]$

The solution in steady state exists, if the conditions $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5<1$ are satisfied.

VI. VARIOUS QUEUE CHARACTERISITCS

A. Average number of customers

L= $\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1+n_2+n_3+n_4+n_5)P_{n_1,n_2,n_3,n_4,n_5}$

$=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1)P_{n_1,n_2,n_3,n_4,n_5}+\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_2)P_{n_1,n_2,n_3,n_4,n_5}$

$+\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_3)P_{n_1,n_2,n_3,n_4,n_5}+\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_4)P_{n_1,n_2,n_3,n_4,n_5}$

$+\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_5)P_{n_1,n_2,n_3,n_4,n_5}$

On Putting the values of $P_{n_1,n_2,n_3,n_4,n_5}, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5$ and after simplication ,

We get

L= $L_1 + L_2 + L_3 + L_4 + L_5$

L= $\frac{\lambda_1+\lambda_2 p_{21}}{(1-p_{21}p_{12})\mu_1-(\lambda_1+\lambda_2 p_{21})} + \frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_2-(\lambda_2+\lambda_1 p_{12})} +$

$\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}}{\mu_3(1-p_{12}p_{21-[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}]})} + \frac{(p_{34}+p_{35}p_{54})(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}}{(\mu_4(1-p_{45}p_{54})(1-p_{21}p_{12}))-((p_{34}+p_{35}p_{54})[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}]} +$
$\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}(p_{45}(p_{34}+p_{35}p_{54})+p_{35})}{(\mu_5(1-p_{45}p_{54})(1-p_{21}p_{12}))-(p_{45}(p_{34}+p_{35}p_{54})+p_{35}[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}]}$

B. *Variance of queue*

V$(n_1+n_2+n_3+n_4+n_5) = \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}[(n_1+n_2+n_3+n_4+n_5)-L]^2 P_{n_1,n_2,n_3,n_4,n_5}$

$= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}[(n_1+n_2+n_3+n_4+n_5)]^2 P_{n_1,n_2,n_3,n_4,n_5} + L^2 \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty} P_{n_1,n_2,n_3,n_4,n_5} - 2L\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}[(n_1+n_2+$
n3+n4+n5Pn1,n2,n3,n4,n5

$=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}[(n_1+n_2+n_3+n_4+n_5)]^2 P_{n_1,n_2,n_3,n_4,n_5} - L^2$

$= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1+n_2+n_3+n_4+n_5)P_{n_1,n_2,n_3,n_4,n_5}$

$=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1)^2 P_{n_1,n_2,n_3,n_4,n_5} + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_2)^2 P_{n_1,n_2,n_3,n_4,n_5} +$
$\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_3)^2 P_{n_1,n_2,n_3,n_4,n_5} + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_4)^2 P_{n_1,n_2,n_3,n_4,n_5} + \sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_5)^2 P_{n_1,n_2,n_3,n_4,n_5} + 2$
$\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1 n_2)P_{n_1,n_2,n_3,n_4,n_5} + 2\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1 n_3)P_{n_1,n_2,n_3,n_4,n_5} +$

$2\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1 n_4)P_{n_1,n_2,n_3,n_4,n_5} +$

$2\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_1 n_5)P_{n_1,n_2,n_3,n_4,n_5} +$

$2\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_2 n_3)P_{n_1,n_2,n_3,n_4,n_5} + \ldots\ldots\ldots\ldots\ldots +$

$2\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\sum_{n_3}^{\infty}\sum_{n_4}^{\infty}\sum_{n_5}^{\infty}(n_4 n_5)P_{n_1,n_2,n_3,n_4,n_5} - L^2$

After Substituting the values ,the result reduces to

$[\frac{\lambda_1+\lambda_2 p_{21}}{(1-p_{21}p_{12})\mu_1}][1-\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_1}]^{-2} + [\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_2}][1-\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_2}]^{-2} + [\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3}][1-\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3}]^{-2}$

$+[\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3}][1-\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3}]^{-2}$

$+[\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})}\right]\left[\frac{p_{45}(p_{34}+p_{35}p_{54})+p_{35}}{\mu_5(1-p_{45}p_{54})}\right]][1-\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})}\right]\left[\frac{p_{45}(p_{34}+p_{35}p_{54})+p_{35}}{\mu_5(1-p_{45}p_{54})}\right]]^{-2}$

C. *Average waiting time for the customers in the system*

Where

E(W)= $\frac{\lambda_1+\lambda_2 p_{21}}{\lambda[(1-p_{21}p_{12})\mu_1-(\lambda_1+\lambda_2 p_{21})]} + \frac{\lambda_2+\lambda_1 p_{12}}{\lambda[(1-p_{21}p_{12})\mu_2-(\lambda_2+\lambda_1 p_{12})]} +$

$\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}}{\lambda[\mu_3(1-p_{12}p_{21-[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}]})]} + \frac{(p_{34}+p_{35}p_{54})(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}}{\lambda[(\mu_4(1-p_{45}p_{54})(1-p_{21}p_{12}))-((p_{34}+p_{35}p_{54})[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}]]} +$
$\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}(p_{45}(p_{34}+p_{35}p_{54})+p_{35})}{\lambda[(\mu_5(1-p_{45}p_{54})(1-p_{21}p_{12}))-(p_{45}(p_{34}+p_{35}p_{54})+p_{35}[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_1+\lambda_2 p_{21})p_{13}]]}$

VII. ALGORITHM

The following algorithm provides the procedure to determine the joint probability and various queue characteristics of above discussed queue model:

Step1.Obtain the number of customers $n_1, n_2, n_3, n_4, n_{5.}$

Step2. Obtain the values of mean service rate $\mu_1, \mu_2, \mu_3, \mu_4, \mu_{5.}$

Step3.Obtain the values of mean arrival rate λ_1, λ_2.

Step4.Obtain the values of the probabilities $p_{12}, p_{21}, p_{13}, p_{23}, p_{34}, p_{35}, p_{45}, p_{46}, p_{54}, p_{57.}$

Step5. Calculate the value of

(i) $F_1 = 1 - \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{21} p_{12})\mu_1}$

(ii) $F_2 = 1 - \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{21} p_{12})\mu_2}$

(iii) $F_3 = 1 - \frac{[(\lambda_2 + \lambda_1 p_{12})p_{23} + (\lambda_2 + \lambda_1 p_{12})p_{13}]}{(1 - p_{21} p_{12})\mu_3}$

(iv) $F_4 = 1 - \left[(p_{34} + p_{35} p_{54})\left[\frac{(\lambda_2 + \lambda_1 p_{12})p_{23} + (\lambda_2 + \lambda_1 p_{12})p_{13}]}{(1 - p_{21} p_{12})\mu_4 (1 - p_{45} p_{54})}\right]\right]$

(v) $F_5 = 1 - \left[\frac{(\lambda_2 + \lambda_1 p_{12})p_{23} + (\lambda_2 + \lambda_1 p_{12})p_{13}]}{(1 - p_{21} p_{12})}\right]\left[\frac{p_{45}(p_{34} + p_{35} p_{54}) + p_{35}}{\mu_5 (1 - p_{45} p_{54})}\right]$

Step 6.Calculate

(i) $\rho_1 = 1 - F_1$

(ii) $\rho_2 = 1 - F_2$

(iii) $\rho_3 = 1 - F_3$

(iv) $\rho_4 = 1 - F_4$

(v) $\rho_5 = 1 - F_5$

Step 7.Check $\rho_1, \rho_2, \rho_3, \rho_4, \rho_{5.} < 1$

If so, then go to step (8) else steady state condition does not holds good.

Step 8. The joint probability

$$P_{n_1, n_2, n_3, n_4, n_5} = \rho^{n_1} \rho^{n_2} \rho^{n_3} \rho^{n_4} \rho^{n_5} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4)(1 - \rho_5)$$

Step9. Calculate average no. of customers (mean queue length)

$$L = \frac{\rho_1}{(1-\rho_1)} + \frac{\rho_2}{(1-\rho_2)} + \frac{\rho_3}{(1-\rho_3)} + \frac{\rho_4}{(1-\rho_4)} + \frac{\rho_5}{(1-\rho_5)}$$

Step10. Calculate variance of queue

$$V = \frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2}$$

Step11.Calculate average waiting time for customers

$E(W)=\frac{L}{\lambda}$.

VIII. NUMERICAL ILLUSTRATION

Given customers coming to three servers out of which two servers consists biserial channels and one server is commonly linked in series with each of the two servers in biseries. The number of customers, mean service rate, mean arrival rate and associated probabilities are given as follows:

S.No.	No. of Customers	Mean Service Rate	Mean Arrival Rate	Probabilities
1.	n_1=3	μ_1=9	λ_1=2	p_{12}=0.6
2.	n_2=6	μ_2=8	λ_2=3	p_{13}=0.4
3.	n_3=7	μ_3=7		p_{21}=0.7
4.	n_4=4	μ_4=10		p_{23}=0.3
5.	n_5=5	μ_5=15		p_{45}=0.5
				p_{46}=0.5
				p_{34}=0.6
				p_{35}=0.4
				p_{54}=0.7
				p_{57}=0.3

SOLUTION:

$$\rho_1=\frac{\lambda_1+\lambda_2 p_{21}}{(1-p_{21}p_{12})\mu_1} =0.78$$

$$\rho_2=\frac{\lambda_2+\lambda_1 p_{12}}{(1-p_{21}p_{12})\mu_2} =0.90$$

$$\rho_3=\frac{[(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_3} =0.71$$

$$\rho_4=\left[(p_{34}+p_{35}p_{54})\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})\mu_4(1-p_{45}p_{54})}\right]\right] =0.68$$

$$\rho_5=\left[\frac{(\lambda_2+\lambda_1 p_{12})p_{23}+(\lambda_2+\lambda_1 p_{12})p_{13}]}{(1-p_{21}p_{12})}\right]\left[\frac{p_{45}(p_{34}+p_{35}p_{54})+p_{35}}{\mu_5(1-p_{45}p_{54})}\right] =0.25$$

Average no. of customers

$L= L_1+L_2+L_3+L_4+L_5$

= 3.66+9.54+2.5+2.50+0.8193

= 19.0193

Average waiting time for customers

$E(W)=\frac{L}{\lambda}$ where $\lambda = \lambda_1 + \lambda_2$

$=\frac{L_1+L_2+L_3+L_4+L_5}{\lambda}$

= 0.732+1.908+0.5+0.5+0.16386

= 3.80386

Variance- $\frac{\rho_1}{(1-\rho_1)^2}+\frac{\rho_2}{(1-\rho_2)^2}+\frac{\rho_3}{(1-\rho_3)^2}\frac{\rho_4}{(1-\rho_4)^2}+\frac{\rho_5}{(1-\rho_5)^2}$

= 0.037752+90+8.442+6.6406+0.44

= 105.560352

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