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Modulation of Generalized Canonical CS-Transform

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ABSTRACT

In this paper the generalized canonical CS- transform it is extended to the distribution of compact support by using kernel method. The analyticity and modulation theorems are proved for this transform. Also some important results discussed about the kernel of CS-transform.

Keywords: Generalized function,2D Canonical transform, Canonical CS-transforms, Modulation theorem,2D Fractional Fourier transform.

2000MSC No: 44-XX, 26A-43, 42-38, 46F-12, 42-XX.

1 INTRODUCTION

As generalization of the fractional Fourier transform, the linear canonical transform has been used in several areas, including optics and signal processing, image reconstruction [1], [2], [3]. The definition canonical CS- transform is as follows,

$$\{CCST\ f(t,x)\}(s,w)=\langle f(t,x),K_c(t,s)K_s(x,w)\rangle$$

Notation and terminology of this paper is as per Zemanian [4], [5].The paper is organized as follows. Section 2 gives the definition of canonical CS- transform on the space of generalized function and states the properties of the kernel of the canonical CS- transform. In section 3 modulation theorems are proved. In section 4 analyticity theorem is proved and lastly the conclusion is stated.

2. DEFINITION OF GENERALIZED CS- TRANSFORM:

Let $E'(R\times R)$ denote the dual of $E(R\times R)$ .Therefore the generalized canonical CS-transform of $f(t,x)\in E'(R\times R)$ is defined as

$$\{CCST\ f(t,x)\}(s,w)=\langle f(t,x),K_c(t,s)K_s(x,w)\rangle$$

$$\therefore \{CCST\ f(t,x)\}(s,w)$$

$$=-i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dx\,dt$$

Where

$$K_c(t,s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos\left(\frac{s}{b}t\right)$$

when $b\neq 0$

$$= \sqrt{d}\, e^{\frac{i}{2}\left(cds^2\right)} \delta(t-ds) \qquad \text{when b = 0}$$

and $$K_s(x,w) = (-i)\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} \sin\left(\frac{s}{b}x\right)$$

when $b \neq 0$

$$= \sqrt{d}\, e^{\frac{i}{2}\left(cdw^2\right)} \delta(x-dw) \qquad \text{when b = 0}$$

where

$$\gamma_{E,k}\left\{K_c(t,x)K_s(x,w)\right\} = \sup_{\substack{-\infty<t<\infty \\ -\infty<x<\infty}} \left|D_t^k D_x^l K_c(t,x)K_s(x,w)\right| < \infty$$

2.1 PROPERTIES OF KERNEL

Kernel of CS-transform satisfied following properties.

(a) $K_{C_1}(t,s)\, K_{S_1}(x,w) = K_{C_1}(s,t)\, K_{S_2}(w,x)$ if $a=d$

(b) $K_{C_1}(t,s)\, K_{S_1}(x,w) \neq K_{C_1}(s,t)\, K_{S_1}(w,x)$ if $a \neq d$

(c) $K_{C_1}(-t,s)\, K_{S_1}(x,w) = K_{C_1}(t-s)\, K_{S_1}(x,w)$

(d) $K_{C_1}(-t,s)\, K_{S_1}(-x,w) = K_{C_1}(t,s)\, K_{S_1}(x,w)$

Above properties of kernel are simple to prove.

3. MODULATION THEOREM FOR CANONICAL CS-TRANSFORM:

Theorem 3.1: If $\{CCST\, f(t,x)\}(s,w)$ is canonical CS- transform of $f(t,x)$ then

$$\{CCST \cos \mu t\, f(t,x)\}(s,w) = \frac{e^{-\frac{i}{2}\left(\mu^2 bd\right)}}{2}\left[e^{-i(s\mu d)}\{CCST\, f(t,x)\}(s+\mu b, w) + e^{i(\mu s d)}\{CCST\, f(t,x)\}(s-\mu b, w)\right]$$

Proof: Definition of canonical CS-transform $f(t,x)$ is

$$\{CCST\, f(t,x)\}(s,w) = -i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt$$

$$\{CCST \cos \mu t\, f(t,x)\}(s,w) = -i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)\cos \mu t\, e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt$$

$$= -i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\cos\left(\frac{s}{b}t\right)\cos(\mu t)\right]\sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt$$

$$= \frac{1}{2}\left[-i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\left(\frac{s+\mu b}{b}\right)t \sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt\right.$$

$$\left. -i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos\left(\frac{s-\mu b}{b}\right)t \sin\left(\frac{w}{b}x\right) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt\right]$$

$$= \frac{1}{2}\left[e^{-i(s\mu d)} e^{\frac{-i}{2}\left(u^2 bd\right)}\{CCST\, f(t,x)\}(s+\mu b, w) + e^{i(s\mu d)} e^{\frac{-i}{2}\left(\mu^2 bd\right)}\{CCST\, f(t,x)\}(s-\mu b, w)\right]$$

$$\{CCST \cos \mu t\, f(t,x)\}(s,w)$$

$$= \frac{e^{-\frac{i}{2}\left(\mu^2 bd\right)}}{2}\left[e^{-i(s\mu d)}\{CCST\, f(t,x)\}(s+\mu b, w) + e^{i(\mu s d)}\{CCST\, f(t,x)\}(s-\mu b, w)\right]$$

Theorem 3.2: If $\{CCST\, f(t,x)\}(s,w)$ is canonical CS-transform of $f(t,x)$ then

$$\{CCST \sin \mu t\, f(t,x)\}(s,w)$$

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$$= \frac{ie^{-\frac{i}{2}(\mu^2 bd)}}{2}\left[e^{-i(s\mu d)}\{CSST\ f(t,x)\}(s+\mu b,w)-e^{i(\mu sd)}\{CSST\ f(t,x)\}(s-\mu b,w)\right]$$

Proof: Definition of canonical CS-transform $f(t,x)$ is

$$\{CCST\ f(t,x)\}(s,w)=$$
$$-i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt$$

$$\{CCST\sin\mu t\ f(t,x)\}(s,w)=$$
$$-i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\left(\frac{s}{b}t\right)\sin\left(\frac{w}{b}x\right)\sin\mu t\, e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt$$

$$=-i\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\cos\left(\frac{s}{b}t\right)\sin(\mu t)\right]\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt$$

$$=\frac{i}{2}\left[-\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s+\mu b}{b}\right)t\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt\right.$$
$$\left.\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin\left(\frac{s-\mu b}{b}\right)t\sin\left(\frac{w}{b}x\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f(t,x)\,dxdt\right]$$

$$=\frac{i}{2}\left[e^{-i(s\mu d)}e^{\frac{-i}{2}(u^2bd)}i\{CSST\ f(t,x)\}(s+ub,w)\right.$$
$$\left.-e^{i(s\mu d)}e^{\frac{-i}{2}(\mu^2bd)}\{CSST\ f(t,x)\}(s-ub,w)\right]$$

$$\{CSST\sin\mu t\ f(t,x)\}(s,w)$$

$$=\frac{ie^{-\frac{i}{2}(\mu^2bd)}}{2}\left[\begin{array}{l}e^{-i(s\mu d)}\{CSST\ f(t,x)\}(s+\mu b,w)\\-e^{i(\mu sd)}\{CSST\ f(t,x)\}(s-\mu b,w)\end{array}\right]$$

Theorem 3.3: If $\{CCST\ f(t,x)\}(s,w)$ is canonical CS- transform of $f(t,x)$ then

$$\{CCSTe^{i\mu t}\ f(t,x)\}(s,w)=$$
$$\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\left(\{CCST\ f(t,x)\}(s+\mu b,w)-\{CSST\ f(t,x)\}(s+\mu b,w)\right)\right.$$
$$\left.+e^{i(s\mu d)}\left(\{CCST\ f(t,x)\}(s-\mu b,w)+\{CSST\ f(t,x)\}(s-\mu b,w)\right)\right]$$

Proof: Since

$$\{CCST\,e^{i\mu t}\ f(t,x)\}(s,w)$$
$$=\{CCST(\cos\mu t+i\sin\mu t)\,f(t,x)\}(s,w)$$

$$\{CCST\,e^{i\mu t}\ f(t,x)\}(s,w)=$$
$$\{CCST\cos\mu t\ f(t,x)\}(s,w)+i\{CCST\sin\mu t\,f(t,x)\}(s,w)$$

Now using cosine modulation sine modulation

$$\{CCSTe^{i\mu t}\ f(t,x)\}(s,w)=$$
$$\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\{CCST\ f(t,x)\}(s+\mu b,w)+e^{i(s\mu d)}\{CCST\ f(t,x)\}(s-\mu b,w)\right]$$

$$-\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\{CSST\ f(t,x)\}(s+\mu b,w)-e^{i(s\mu d)}\{CSST\ f(t,x)\}(s-\mu b,w)\right]$$

$$\{CCSTe^{i\mu t}\ f(t,x)\}(s,w)$$
$$=\frac{e^{-\frac{i}{2}(\mu^2bd)}}{2}\left[e^{-i(s\mu d)}\left(\{CCST\ f(t,x)\}(s+\mu b,w)-\{CSST\ f(t,x)\}(s+\mu b,w)\right)\right.$$
$$\left.+e^{i(s\mu d)}\left(\{CCST\ f(t,x)\}(s-\mu b,w)+\{CSST\ f(t,x)\}(s-\mu b,w)\right)\right]$$

4. ANALYTICITY THEOREM:

Theorem4.1.(Analyticity) Let $f\in\ E^1(R^n)$ and its canonical CS-transform be defined by,

$$\{CCST\ f(t,x)\}(s,w)=$$
$$-i\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}\cos\left(\frac{s}{b}\right)t\sin\left(\frac{w}{b}\right)x\ f(t,x)\,dxdt$$

then $\{CCST\ f(t,x)\}(s,w)$ is analytic on C^n, if the a, b, $\sup pf \subset s_a$ and s_b where

$$s_a = \{t : t \in R^n, \quad |t| \le a, a > 0\},$$
$$s_b = \{x : x \in R^n, \quad |x| \le b, b > 0\}$$ moreover

$\{CCST\ f(t,x)\}(s,w)$ is differentiable and

$$D_s^k D_w^l \{CCST\ f(t,x)\}(s,w) = \left\langle f(t,x), D_s^k D_w^l K_{C_1}(t,s) K_{S_1}(x,w) \right\rangle$$

Proof: Let, $s: \{s_1, s_2, \dots\dots s_j \dots\dots s_n\} \in C^n$ and $w:\{w_1, w_2, \dots\dots w_j \dots\dots w_n\} \in C^n$

We first prove that,

$\frac{\partial}{\partial s_j} \frac{\partial}{\partial w_j} \{CCST\ f(t,x)\}(s,w)$ exists,

$$\frac{\partial^n}{\partial s_j^n} \frac{\partial^n}{\partial w_j^n} \{CCST\ f(t,x)\}(s,w)$$
$$= \left\langle f(t,x), \frac{\partial^n}{\partial s_j^n} \frac{\partial^n}{\partial w_j^n} K_{c_1}(t,s) K_{s_1}(x,w) \right\rangle \qquad (4.1)$$

we prove the result $n = 1$, the general result following by induction.

For fixed $s_j \neq 0$ choose two concentric circles C and C^1 with centre s_j and radii r and r_1 respectively

such that $0<r<r_1< |s_j|$. Let Δs_j be a complex increment satisfying $0 < |\Delta s_j| < r$. Also for fixed

$w_j \neq 0$. Again choose two concentric circles C and C_1 with centre w_j and radii r' and r_1' respectively

such that $0 < r' < r_1' < |w_j|$. Let Δw_j be a complex increment satisfying $0 < |\Delta w_j| < r'$

$$\frac{(CCST)(s_j + \Delta s_j, w_j) - (CCST)(s_j, w_j)}{\Delta s_j} \cdot$$
$$\frac{(CCST)(s_j, w_j + \Delta w_j) - (CCST)(s_j, w_j)}{\Delta w_j}$$
$$- < f(t,x), \frac{\partial}{\partial s_j} \frac{\partial}{\partial w_j} K_{c_1}(t,s) K_{s_1}(x,w) >$$
$$= \left\langle f(t,x), \Psi \Delta s_j(t) \Psi \Delta w_j(x) \right\rangle \qquad (4.2)$$

where

$$\Psi \Delta s_j(t) \Delta w_j(x) = \frac{1}{\Delta s_j} \left[K_{c_1}(t, s_1, s_2 \cdots\cdots s_j + \Delta s_j \cdots s_n) - K_{c_1}(t,s) \right]$$

$$\frac{1}{\Delta w_j} \left[K_{s_1}(x, w_1, w_2 \cdots\cdots w_j + \Delta w_j \cdots w_n) - K_{s_1}(x,w) \right]$$
$$- \frac{\partial^n}{\partial s_j^n} \frac{\partial^n}{\partial w_j^n} K_{c_1}(t,s) K_{s_1}(x,w)$$

For any fixed $(t,x) \in R^n$ and any fixed integer. $k = (k_1, k_2 \cdots\cdots k_n) \in N_0$ and $l = (l_1, l_2 \cdots\cdots l_n) \in N_0$

$D_t^k D_x^l K_{c_1}(t,s) K_{s_1}(x,w)$ is analytic inside and on C' and C_1'.

By Cauchy integral formula.

$$D_t^k D_x^l \ \Psi \Delta s_j \Delta w_j(t,x)$$
$$= \frac{1}{4\pi^2 i^2} D_t^k D_x^l \int_{c} \int_{c_1} K_{c_1}(t,s) K_{s_1}(x,w) \left(\frac{1}{\Delta s_j} \left(\frac{1}{z - s_j - \Delta s_j} - \frac{1}{z - s_j} \right) - \frac{1}{(z - s_j)^2} \right)$$
$$\left(\frac{1}{\Delta w_j} \left(\frac{1}{y - w_j - \Delta w_j} - \frac{1}{y - w_j} \right) - \frac{1}{(y - w_j)^2} \right) dzdy$$

where, $\bar{s} = (s_1 \dots\dots s_{j-1}, z, s_{j+1} \dots\dots s_n)$ and $\bar{w} = (w_1 \dots\dots w_{j-1}, y, w_{j+1} \dots\dots w_n)$.

$$=\frac{\Delta s_j \Delta w_j}{-4\pi^2}\int_{c'}\int_{c_1'}\frac{D_t^k D_x^l\, K_{c_1}(t,\bar{s})K_{s_2}(x,\bar{w})}{(z-s_j-\Delta s_j)\,(z-s_j)^2(y-w_j-\Delta w_j)\,(y-w_j)^2}\,dzdy,$$

But for all $z\in C'$ and $y\in C_1'$ and (t,x) restricted to a compact subset of R^n,

$D_t^k\, D_x^l K_{c_1}(t,s)K_{s_1}(x,w)$ is bounded by constant Q.

$$\left|D_t^k\, D_x^l \Psi\Delta s_j \Delta w_j(t,x)\right|\le\frac{|\Delta s_j||\Delta w_j|}{4\pi^2}\int_{c'}\int_{c_1'}\frac{Q}{(r_1-r)\,(r_1)(r_1'-r)\,(r_1')}|dz||dy|$$

$$\le\frac{|\Delta s_j||\Delta w_j|}{4\pi^2}\,\frac{Q}{(r_1-r)\,(r_1)(r_1'-r)\,(r_1')}$$

Thus as $|\Delta s_j|\to 0,$ and $|\Delta w_j|\to 0,$ $D_t^k\, D_x^l \Psi\Delta s_j \Delta w_j(t,x)$ tends to zero uniformly on the compact subset of R^n, therefore it follows that $\Psi\Delta s_j \Delta w_j(t,x)$ converges in $E(R^n)$ to zero. Since $f\in E^1$, we conclude (4.2) tends to zero. Therefore $\{CCST\ f(t,x)\}(s,w)$ is differentiable with respective s_j and w_j. But this is true for all i, $j=1,2,\ldots\ldots n$ .Hence by induction method we can easily show that $\{CCSTf(t,x)\}(s,w)$ is infinitely differentiable

$\{CCSTf(t,x)\}(s,w)$ is analytic on C^n.and $D_s^k D_w^l\{CCST\ f(t,x)\}(s,w)=\left\langle f(t,x), D_s^k D_w^l\, K_{c_1}(t,s)K_{s_1}(x,w)\right\rangle$
.

6 CONCLUSIONS:

The canonical CS-transform, which is the generalization of number of transforms is itself generalized to the spaces of generalized functions as per Zemanian. This transform is used to solve ordinary or partial differential equations. This transform is an important tool in signal processing and many other branches of engineering. In this paper discussed some properties of CS-transform.

REFERANCES:

[1] Almeida L.B., "The fractional Fourier transform and time frequency representation IEEE trans on signal processing",Vol. 42, No.11 (1994).

[2] Alieva Tatiana. and Bastiaans Martin J., "Signal Reconstruction from Two Close Fractional Fourier Power Spectra", IEEE Trans.On Signal Processing, Vol.51, No.1, Jan.(2003)

[3] Namias V., "The fractional order Fourier transform and its applications in quantum mechanics", I. Inst. Maths. appli., 25, 241-265 (1980).

[4] Zemanian A.H., "Distribution theory and transform analysis", McGraw Hill, New York, (1965)

[5] Zemanian A.H., "Generalized Integral Transform", Inter Science Publisher's New York, (1968).