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The dynamics of disease transmission in a Prey Predator System with harvesting of prey

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***Abstract*:** In this present work on predator-prey system, a disease transmission model, incorporating provisions for harvesting of prey have been discussed and analyzed. The disease spread is from infected prey to predator species. The SI models are considered for both the species. Conditions are derived for existence and stability of disease free equilibrium state of the system. The effect of harvesting of prey has been analyzed from point view of the stability of system. The epidemiological threshold R_0 and R_1 quantities have been obtained using next generation approach for the model system. Conditions for endemic disease in prey species are discussed. This study also considers the impact of disease in prey on the extinction aspect of the predator. Numerical and computer simulations of the model have been done.

Mathematics Subject Classification (2000) 35B35 - 92C05 - 92D30 -92D40

Keywords: Predator, prey, Equilibrium point, carrying capacity, Basic reproduction number, Stability, Harvesting

1. Introduction:

Predator-prey interactions give regulatory system for the advancement of biological species. in natural ecosystem. Prey predator interactions and disease in the system affects each other [1, 3, 5, 7, 8-10, and 17]. Some previous studies of infectious diseases in animal populations were focused on the effects of disease-induced mortality or disease-reduced reproduction in the regulation of populations in their natural habitats [3, 7, 8, 17 and 18]. Reduced population sizes and destabilization of equilibrium into oscillations are caused by the presence of infectious disease in one or both of the populations.

The fact that predators take a disproportionate number of infected preys has been confirmed by earlier studies [8, 11 and 17]. Infected prey is more exposed to predation on account of the fact that the disease may weaken their ability to protect themselves and expose them to predators [1]. This increased vulnerability of infected prey may lead to persistence of the predator (which would otherwise be starved and become extinct). Further, predation of the infected prey may lead to the disappearance of disease which may otherwise be endemic in prey population [3]. Together with no reproduction in infected prey, as predation on infected prey increases, the infection tends to destabilize prey-predator interactions [17].

Most of the eco-epidemiological studies are restricted to the situations where only prey species can get infection. Few investigations [4, 11, 19-21] consider the spread of disease from prey to predator through predation of infected prey. For the purpose of eliminating either the disease from predator- prey system or eliminating the menace of the predator by controlling disease in prey, a simplified eco- epidemiological four species prey predator SI model [4] is shown to be useful.

Exploitation of biological resources as practiced in fishery, and forestry has strong impact on dynamic evolution of biological population. The over exploitation of resources may lead to extinction of species which adversely affects the ecosystem. However, reasonable and controlled harvesting is beneficial from economical and ecological point of view. The research on harvesting in predator-prey systems has been of interest to economists, ecologists and natural resource management for some time now. Very few has explicitly put a harvested parameter in a predator–prey–parasite model and studied its effect on the system. Here we study the role of harvesting in an eco-epidemiological system where the susceptible and infected prey are subjected to combined harvesting.

In this present work on predator-prey system, a disease transmission model, incorporating provisions for harvesting of prey have been proposed and studied. In the model, the disease is spreading from infected prey to susceptible prey and consequently to the predator species via predation related activities: capturing, handling and eating the infected prey. Both, the prey and the predator species are compartmentalized into susceptible and infected classes. Recovery from disease is not considered for both prey and the predator species. Consequently, SI model is considered for both the species.

2. Mathematical Analysis

Consider an eco-epidemiological system comprising of four populations, namely, the susceptible prey (S_1), infected prey (I_1), susceptible predator (S_2) and infective predator (I_2)

such that $N_1(t) = S_1(t) + I_1(t)$ and $N_2(t) = S_2(t) + I_2(t)$. An SI type eco-epidemiological model for prey predator dynamics is presented under following assumptions:

Only susceptible prey is capable of reproducing logistically. While susceptible prey can easily escape the predation, the infected prey are weakened due to disease and become easier to catch. The infected prey and predator do not reproduce but still continue to consume resources. The susceptible predator gets food from infected prey for its survival. It may become infected due to interaction with infected prey. Mortality rate of infected predator is higher than that of susceptible predator. It is also assumed that the disease does not affect the hunting ability of infected predator. Further, all the predators who predate the infected prey become infected. Let e_1 be the harvesting rate of susceptible prey and e_2 be the harvesting rate of infected prey. Since it is easy to catch infected prey as compare to susceptible prey, therefore $e_1 < e_2$.

Accordingly, the following mathematical model is obtained under the above assumptions:

$$\frac{dS_1}{dt} = rS_1\left(1 - \frac{S_1 + I_1}{K_1}\right) - a_1 S_1 (S_2 + I_2) - c_1 S_1 I_1 - e_1 S_1$$

$$\frac{dI_1}{dt} = c_1 S_1 I_1 - a_2 I_1 (S_2 + I_2) - d_1 I_1 - e_2 I_1$$

$$\frac{dS_2}{dt} = -a_2 S_2 I_1 + (-d_2 + k a_1 S_1) S_2$$

$$\frac{dI_2}{dt} = a_2 S_2 I_1 - d_3 I_2 \tag{1}$$

where

$a_1 =$ Predation rate on the susceptible prey (hectare per individual day^{-1})

$a_2 =$ Predation rate on the infective prey (hectare per individual day^{-1})

$k =$ Feeding efficiency of predator (per individual day^{-1})

$r =$ Intrinsic growth rate constant of susceptible prey (day^{-1})

$K_1 =$ Carrying capacity of environment with respect to prey species (Individual ha^{-1})

$c_1 =$ Rate of force of infection (hectare per individual day^{-1})

$d_1 =$ Net death rate of prey (Natural death + death due to infection) (day^{-1})

$d_2 =$ Natural death rate of susceptible predator (day^{-1})

$d_3 =$ Net death rate of infected predator (day^{-1})

According to the assumptions $a_2 > a_1$, $d_3 > d_2$ and $e_2 > e_1$, $k < 1$. Also, $S_1(0) \ge 0, S_2(0) \ge 0, I_1(0) \ge 0$ and $I_2(0) \ge 0$.

3. Mathematical Analysis

Theorem 1 All the solutions of the system (1) which initiate in $\mathbb{R}_+^4$ are uniformly bounded.

Proof: By theorems of Nagumo [13] , it can be easily proved that S_1, I_1, S_2 and I_2 remain positive. Let $W_1 = S_1 + I_1$

From (1), the following can be easily obtained:

$$\frac{d(S_1+I_1)}{dt} \le rS_1\left(1 - \frac{S_1}{K_1}\right) - e_1S_1 - d_1I_1 - e_2I_1$$

$$\frac{d(S_1+I_1)}{dt} \le (r-e_1)S_1 - \frac{rS_1^2}{K_1} - (d_1+e_2)I_1$$

For arbitrarily chosen η_1, this simplifies to

$$\frac{d(S_1+I_1)}{dt} + \eta_1(S_1+I_1) \le (r-e_1+\eta_1)S_1 - \frac{rS_1^2}{K_1} - (d_1+e_2-\eta_1)I_1$$

$$\frac{dW_1}{dt} + \eta_1W_1 \le \frac{K_1(r-e_1+\eta_1)^2}{4r} - \frac{r}{K_1}\left(S_1^2 - (r-e_1+\eta_1)\frac{K_1}{r}S_1 + \frac{K_1^2(r-e_1+\eta_1)^2}{4r^2}\right) - (d_1+e_2-\eta_1)I_1$$

$$\frac{dW_1}{dt} + \eta_1W_1 \le \frac{K_1(r-e_1+\eta_1)^2}{4r} - \frac{r}{K_1}\left(S_1 - (r-e_1+\eta_1)\frac{K_1}{2r}\right)^2 - (d_1+e_2-\eta_1)I_1$$

Choosing $\eta_1 < d_1 + e_2$ and applying the results of Birkhoff and Rota [6], yields

$$0 < W_1(S_1(t), I_1(t)) \le \frac{L_1}{\eta_1}(1 - e^{-\eta_1 t}) + W_1(S_1(0), I_1(0))e^{-\eta_1 t};\ L_1 = \frac{K_1(r-e_1+\eta_1)^2}{4r}$$

As $t \to \infty$, it gives

$$0 < S_1 + I_1 \le \frac{L_1}{\eta_1}(=K),$$

$$\Rightarrow W_1 = S_1 + I_1 \text{ is bounded}$$

From positivity of S_1 and I_1,

$$0 \le S_1 \le K\ ; 0 \le I_1 \le K.$$

Similarly, defining another positive definite function W as

$$W = S_1 + I_1 + S_2 + I_2$$

Proceeding in the similar manner and choosing $\eta = \min(d_1 + e_2, d_2, d_3)$ yields

$$0 < W(S_1, I_1, S_2.I_2) \le \frac{L_1(r+\eta)}{\eta\eta_1}(1-e^{-\eta t}) + W(S_1(0), I_1(0), S_2(0), I_2(0))e^{-\eta t}$$

As $t \to \infty$, it gives

$$0 < W \le \frac{K(r+\eta)}{\eta} = L \text{, where } L = \frac{(r+\eta)}{\eta}$$

$$\Rightarrow W = S_1 + I_1 + S_2 + I_2 \text{ is bounded.}$$

Hence all solutions of (1) that initiate in $\mathbb{R}_+^4$ are confined in the region

$$B = \left\{(S_1, I_1, S_2, I_2) \in \mathbb{R}_+^n : S_i \le L;\ I_i \le L;\ \ i = 1,2\right\}$$

4. Equilibrium points:

The following equilibrium points exist for the system (1):

a) The trivial equilibrium point $E_0(0,0,0,0)$

b) The axial equilibrium point $E_1(S_1, 0, 0, 0)$,

Where $S_1 = \left(\frac{K_1(r-e_1)}{r}\right)$ always exist provided

$$r > e_1. \tag{2}$$

c) The planar equilibrium point $E_2(S_1^{'}, I_1^{'}, 0, 0)$ in $S_1 - I_1$ plane is obtained as

$$S_1^{'} = \frac{d_1 + e_2}{c_1},\ I_1^{'} = \frac{\{c_1 K_1 (r - e_1) - (d_1 + e_2) r\}}{c_1(c_1 K_1 + r)},$$

The existence of this equilibrium point is possible provided

$$c_1 K_1 (r - e_1) - (d_1 + e_2) r > 0 \tag{3}$$

In this case, the predator species is eliminated and the disease persists in the prey species. Harvesting of susceptible and infected prey will reduce the possibility of existence of

$E_2(S_1^{'}, I_1^{'}, 0, 0)$. If it exist then number of susceptible prey will increase but number of infected prey will decrease due to harvesting of susceptible and infected prey.

Another disease free planar equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ on $S_1 - S_2$ plane exists provided

$$ka_1K_1(r - e_1) - rd_2 > 0 \qquad (4)$$

where

$$\hat{S}_1 = \frac{d_2}{ka_1} \text{ and, } \hat{S}_2 = \frac{ka_1K_1(r - e_1) - rd_2}{ka_1^2K_1},$$

It has been also observed that even if the feeding efficiency k of predator is high enough that $K_1 > \frac{d_2}{ka_1}$, even then the disease free equilibrium in $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ may not exist with harvesting of susceptible prey. If it exists then number of susceptible prey will remain same but number of susceptible predator will decrease due to removal of susceptible prey.

d) For non-zero equilibrium point $E^*(S_1^*, I_1^*, S_1^*, I_1^*)$ of the system (1) is given by

$$S_1^* = \frac{(r - e_1)a_2K_1 + a_1c_1K_1S_1^{'} + (r + c_1K_1)ka_1\hat{S}_1}{r(a_2 + ka_1) + a_1c_1K_1(1 + k)}$$

$$I_1^* = \frac{ka_1(S_1^* - \hat{S}_1)}{a_2}, \qquad S_1^* + I_1^* < K_1$$

$$S_2^* = \frac{c_1d_3(S_1^* - S_1^{'})}{a_2(d_3 + (S_1^* - \hat{S}_1)ka_1)},$$

$$I_2^* = \frac{ka_1(S_1^* - \hat{S}_1)S_2^*}{d_3}$$

Accordingly, $E^*(S^*, I^*, S^*, I^*)$ exists provided

$$S_1^* > \max\left(S_1^{'}, \hat{S}_1\right) = \bar{S},\ r > e_1 \qquad (5)$$

The Basic Reproduction Number is defined as the average number of secondary infections when one infective is introduced into a completely susceptible host population. It is also called the basic reproduction ratio or basic reproductive rate. Using next generation approach [2, 15 and 16], the epidemiological threshold quantities for the system (1) are obtained as

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$$R_0 = \frac{c_1 K_1 (r - e_1)}{r(d_1 + e_2)}, \; R_1 = \frac{c_1 \hat{S}_1}{d_1 + e_2 + a_2 \hat{S}_2} \tag{6}$$

Here, R_0 is the basic reproduction number for isolated prey population and R_1 is the basic reproduction in the prey population when both prey and predator are present in the system. The greater vulnerability of prey will reduce R_1. Harvesting of infected prey (e_2) will reduce both R_1 and R_2. Also by simplification it can be proved that $R_0 > R_1$

5. Local Stability Analysis

The variational matrix V of given system (1) is given by

$$V = \begin{pmatrix} r - \frac{2rS_1}{K_1} - a_1(S_2 + I_2) - c_1 I_1 - \frac{rI_1}{K_1} - e_1 & -\frac{rS_1}{K_1} - c_1 S_1 & -a_1 S_1 & -a_1 S_1 \\ c_1 I_1 & c_1 S_1 - d_1 - a_2(S_2 + I_2) - e_2 & -a_2 I_1 & -a_2 I_1 \\ k a_1 S_2 & -a_2 S_2 & -a_2 I_1 + (-d_2 + k a_1 S_1) & 0 \\ 0 & a_2 S_2 & a_2 I_1 & -d_3 \end{pmatrix}$$

The following theorem is direct consequences of linear stability analysis of the system (1) about E_0 and E_1 .

Theorem 2 The trivial equilibrium point $E_0\,(0,0,0,0)$ is locally asymptotically stable provided

$$r < e_1 \tag{7}$$

Theorem 3 The axial equilibrium $E_1(S_1, 0, 0, 0)$ is locally asymptotically stable provided

$$S_1 < \min\left(S_1^{'}, \hat{S}_1\right) \tag{8}$$

The equilibrium $E_1(S_1, 0, 0, 0)$ is unstable when

$$S_1 \in \left(\min\left(S_1^{'}, \hat{S}_1\right), \infty\right) \tag{9}$$

It can be concluded from (8) that $E_1(S_1, 0, 0, 0)$ may be stable for those parametric values for which it was not stable without harvesting of prey. So harvesting of susceptible and infected prey has positive effect on the stability of $E_1(S_1, 0, 0, 0)$.

Also from (8) it may be noticed that when $E_1(S_1, 0, 0, 0)$ is locally asymptotically stable,

$S_1 < \dfrac{d_1 + e_2}{c_1}$ and $S_1 < \dfrac{d_2}{ka_1}$,

$\Rightarrow K_1 < \dfrac{(d_1 + e_2)r}{c_1(r - e_1)}$ and $K_1 < \dfrac{rd_2}{ka_1(r - e_1)}$ or

$R_0 < 1$ and $K_1 < \dfrac{rd_2}{ka_1(r - e_1)}$

which violate existence conditions (3) and (4) of $E_2(S_1^{'}, I_1^{'}, 0, 0)$ and $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ respectively. So, when $E_1(S_1, 0, 0, 0)$ is locally asymptotically stable then $E_0(0,0,0,0)$, $E_2(S_1^{'}, I_1^{'}, 0, 0)$ and $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ do not exist.

When the perturbations are confined in $S_1 - I_1$ plane only, then the equilibrium point $E_1(S_1, 0, 0, 0)$ is locally asymptotically stable for $R_0 < 1$. However in the presence of predator population the stability depends upon the dynamics of predator also as is evident from (8). In fact, when the feeding efficiency of predator is sufficiently low then the disease will die out from both the prey and predator under condition (8) and the equilibrium point $E_1(S_1, 0, 0, 0)$ is locally asymptotically stable. In other words, when the feeding efficiency is sufficiently low and the basic reproduction number R_0 in prey population is below the threshold then predator population become extinct and disease dies out from the prey population.

Theorem 4. If $R_0 > 1$, then $E_2(S_1^{'}, I_1^{'}, 0, 0)$ locally asymptotically stable provided

$$ka_1 S_1^{'} - a_2 I_1^{'} - d_2 < 0 \qquad (10)$$

Proof: From variational matrix V, the characteristic equation about $E_2(S_1^{'}, I_1^{'}, 0, 0)$ is obtained as

$$\left\{\lambda^2 + \frac{rS_1^{'}}{K_1}\lambda + c_1 I_1^{'}\left(\frac{rS_1^{'}}{K_1} + c_1 S_1^{'}\right)\right\}(-a_2 I_1^{'} + ka_1 S_1^{'} - d_2 - \lambda)\ (-d_3 - \lambda) = 0$$

The quadratic factor gives two eigen values with negative real parts

. Therefore, the system around $E_2(S_1^{'}, I_1^{'}, 0, 0)$ is locally asymptotically stable provided (10) is satisfied. ⊺

If the basic reproduction number R_0 in the prey population is above the threshold, the disease in prey population approaches to endemic level and predator population becomes extinct due to (10).

Harvesting of susceptible and infected prey has negative effect on the stability of $E_2(S_1^{'}, I_1^{'}, 0, 0)$.

The condition of instability of $E_2(S_1^{'}, I_1^{'}, 0, 0)$ if it exists, is given by

$$ka_1S_1^{'} - a_2I_1^{'} - d_2 > 0 \quad (11)$$

Theorem 5. The disease free equilibrium state $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$, if it exists, is locally asymptotically stable provided $R_1 < 1$.

Proof: From variational matrix V, the characteristic equation about $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is

$$(\lambda^2 + \frac{r\hat{S}_1}{K_1}\lambda + ka_1^2\hat{S}_1\hat{S}_2)(c_1\hat{S}_1 - a_2\hat{S}_2 - d_1 - e_2 - \lambda)(-d_3 - \lambda) = 0$$

Since the quadratic factor always yield eigen values with negative real parts, the condition for stability is

$$c_1\hat{S}_1 - a_2\hat{S}_2 - d_1 - e_2 < 0$$

Its simplification gives

$$R_1 < 1 \quad (12)$$

This completes the proof. ↑

When the basic reproduction number R_1 is below the threshold, then the disease dies out from both the species and the equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is stabilized. Rewriting stability condition gives the critical value of e_2 for which the disease free equilibrium state is locally asymptotically stable:

$$e_2 > \left(c_1 + \frac{ra_2}{K_1a_1}\right)\hat{S}_1 + \frac{a_2}{a_1}(r - e_1) - d_1,\ \hat{S}_1 = \frac{d_2}{ka_1} \quad (13)$$

The greater vulnerability of prey to predation may be responsible for persistence of disease free prey predator population.

By simplifying above condition for stability it has been observed that harvesting of infected prey may improve the stability of equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ where as it become less stable with harvesting of susceptible prey. Thus with an appropriate harvesting of susceptible and infected prey, disease free equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ can be maintained at desired level.

The disease free equilibrium state $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ if it exist, is unstable if

$$c_1\hat{S}_1 - a_2\hat{S}_2 - d_1 - e_2 > 0 \tag{14}$$

It is observed that $E_2(S_1^{'}, I_1^{'}, 0, 0)$, if it exists, is locally asymptotically stable provided $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is unstable and vice-versa.

However, it is possible for some choice of parameters that both $E_2(S_1', I_1', 0, 0)$ and $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ exist and locally stable.

It is also observed that when both $E_2(S_1', I_1', 0, 0)$ and $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ are stable then non-zero equilibrium point $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$ always exist.

Theorem 6. The non-zero equilibrium point $E^*(S_1^*, I_1^*, S_1^*, I_1^*)$, if exists, is a saddle point.

Proof: The characteristic equation about E^* is obtained as

$$\lambda^4 + A_0\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{15}$$

Where

$$A_3 = -a_2(d_3 + a_2 I_1^*)\left\{(k+1)a_1c_1 + \left(ka_1 + a_2\right)\frac{r}{K_1}\right\} I_1^* S_1^* S_2^*$$

As the constant term A_3 is always negative, at least one eigenvalue of the equation (15) will be positive, therefore $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$ is not locally asymptotically stable.

It is either unstable or a saddle point. □

6. Numerical simulations

Numerical simulations have been carried out to investigate dynamics of the proposed model. Computer simulations have been performed on MATLAB for different set of parameters.

Consider the following set of parametric values:

$$\begin{aligned} &r = 0.1, K_1 = 50,\ e_1 = 0.03, e_2 = 0.05, d_1 = 0.7,\ d_2 = 0.6, \\ &d_3 = 0.7, c_1 = 0.01, a_1 = 0.8, a_2 = 0.9, k = 0.012 \end{aligned} \tag{16}$$

The system (1) has equilibrium point $E_1(35, 0, 0, 0)$ for the data set (16). It is locally asymptotically stable by the computed value of R_0 is $0.4667 < 1$ and the value of feeding efficiency k of predator population is sufficiently low and the condition (8) is satisfied. The solution trajectories in phase space $S_1 - I_1 - S_2$ are drawn in Fig. 1 for different initial values. All the solution trajectories to tend to $E_1(35,0,0,0)$.

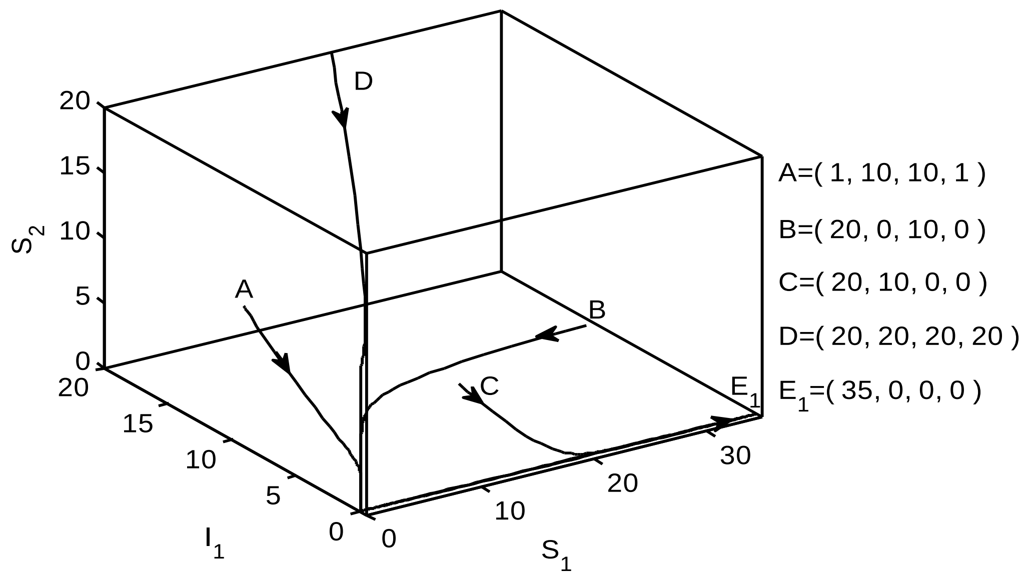

Fig 1: Phase plot depicting the stability of equilibrium point E_1 in phase space S_1 - I_1 - S_2 for data-set (16)

Now consider following set of data

$$r = 0.02, K_1 = 100, e_1 = 0.021, e_2 = 0.03, d_1 = 0.01, d_2 = 0.11,$$
$$d_3 = 0.15, c_1 = 0.002, a_1 = 0.01, a_2 = 0.02, k = 0.33 \quad (17)$$

For above set of data, the equilibrium point $E_0(0,0,0,0)$ is locally asymptotically stable as $r < e_1$ (see Fig. 2) and all other equilibrium points do not exist.

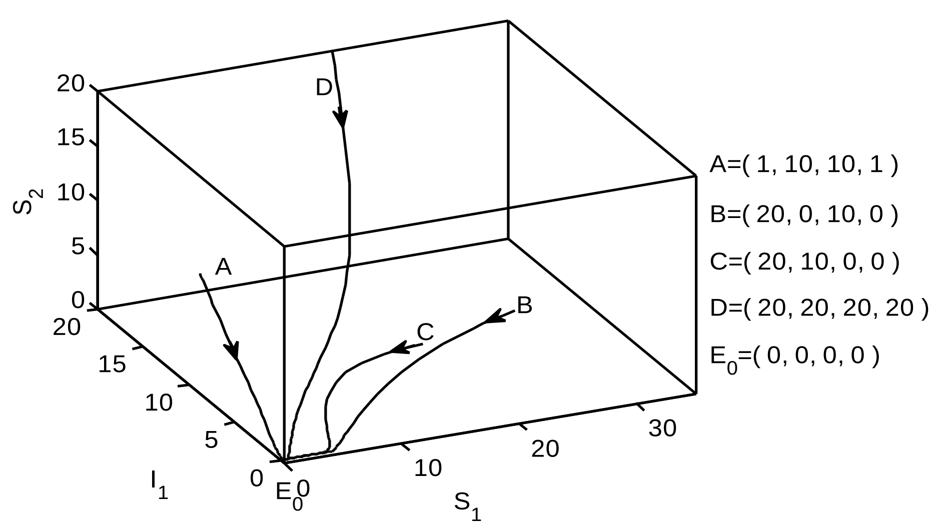

Fig 2: Phase plot depicting the stability of equilibrium point E_1 in phase space S_1 - I_1 - S_2 for data set 17

For the following data set of parameters, E_2 is stable and E_3 is unstable:

$$\begin{aligned} &r = 0.02, K_1 = 100, e_1 = 0.01, e_2 = 0.03, d_1 = 0.01, d_2 = 0.11, \\ &d_3 = 0.15, c_1 = 0.002, a_1 = 0.01, a_2 = 0.02, k = 0.33 \end{aligned} \tag{18}$$

The equilibrium point $E_1(50, 0, 0, 0)$ is unstable as condition (8) is not satisfied and both the planar equilibrium points $E_2(20,\ 2.7273,\ 0,\ 0)$ and $E_3(33.3333,\ 0,\ 0.3333,\ 0)$ exists condition (3) and (4) are satisfied respectively. The equilibrium point $E_3(33.3333,\ 0,\ 0.3333,\ 0)$ is found to be unstable as it satisfies the condition (14). The equilibrium point $E_2(20,\ 2.7273,\ 0,\ 0)$ is found to be locally asymptotically stable as condition (10) is satisfied. The solution trajectories in phase space $S_1 - I_1 - S_2$ are drawn in Fig. 4 for different initial values. All these trajectories converge to $E_2(20,\ 2.7273,\ 0,\ 0)$. Starting with initial value in the neighborhood of the point $E_3(33.3333,\ 0,\ 0.3333,\ 0)$, the solution trajectory approaches to the equilibrium point $E_2(20,\ 2.7273,\ 0,\ 0)$. This confirms the instability of E_3.

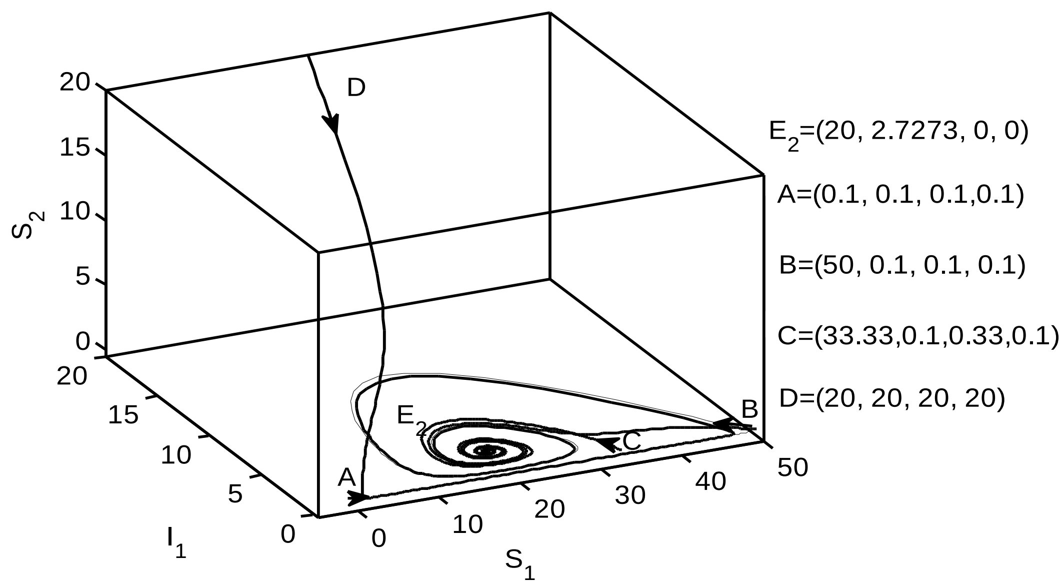

Fig. 3: Phase plot depicting the stability of E_2 and instability of E_3 in $S_1 - I_1 - S_2$ space for dataset (18)

The following data is selected for which all the equilibrium points exist:

$$\begin{aligned} &r = 0.02, K_1 = 100, e_1 = 0.01, e_2 = 0.068, d_1 = 0.01, d_2 = 0.11, \\ &d_3 = 0.15, c_1 = 0.002, a_1 = 0.01, a_2 = 0.02, k = 0.33 \end{aligned} \tag{19}$$

In this case, the equilibrium point $E_1(50,0,0,0)$ is unstable. It is observed that both the equilibrium points $E_2(39, 1, 0, 0)$ and $E_3(33.3333, 0, 0.3333, 0)$ are locally asymptotically stable. The point $E_2(39, 1, 0, 0)$ is stable since $ka_1S_1^{'} - d_3I_1^{'} - d_2 = -0.0013 < 0$ and $R_0 = 1.2821 > 1$. Also, $E_3(33.3333, 0, 0.3333, 0)$ is locally asymptotically stable as $R_1 = 0.7874 < 1$. Both the equilibrium points have their own domains of attractions. Trajectories with different initial conditions converge to different equilibrium points (See Fig. 4).

The instability of non-zero equilibrium point $E^*(39.0915, 0.9501, 0.0081, 0.0010)$ can be seen from Fig. 4. It is also noticed that when even a very small perturbation is taken from $E^*(39.0915, 0.9501, 0.0081, 0.0010)$ to $D(40, 0.9501, 0.0081, 0.0010)$, along S_1 direction, the trajectory converges to $E_2(39, 1, 0, 0)$. Whereas, when the perturbation is taken to $E(39.0915, 1, 0.0081, 0.0010)$, the trajectory of the system converges to $E_3(33.3333, 0, 0.3333, 0)$.

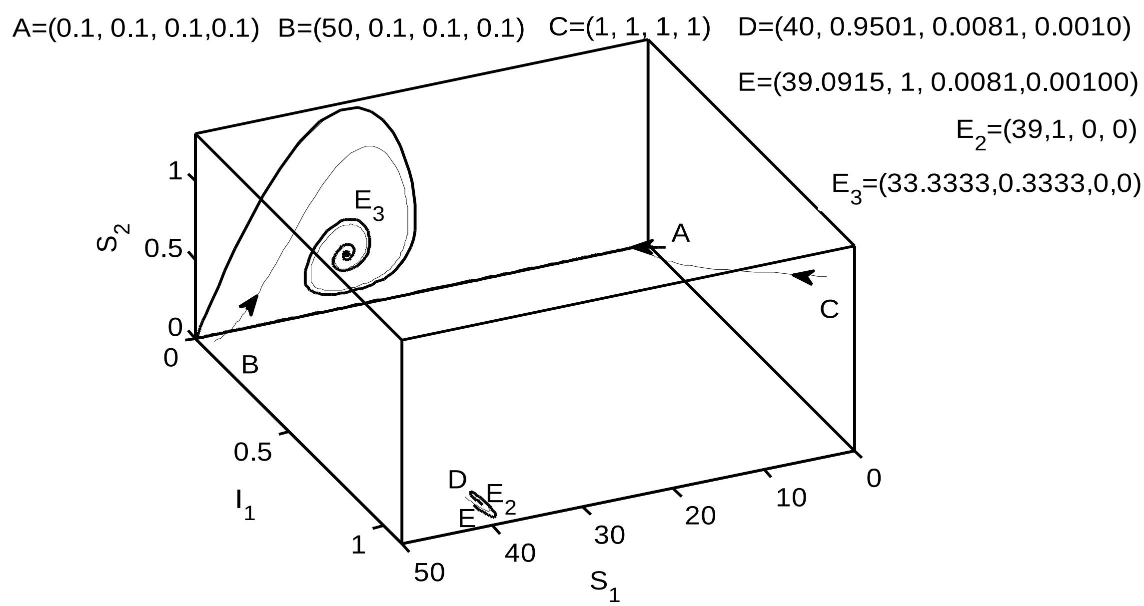

Fig 4: Phase plot depicting the stable behaviour of E_2, E_3 in $S_1 - I_1 - S_2$ space for dataset (19)

Now, consider the following data set where E_3 is stable and E_2 unstable:

$$\begin{aligned} &r = 0.02, K_1 = 100, e_1 = 0.01, e_2 = 0.069, d_1 = 0.01, d_2 = 0.11, \\ &d_3 = 0.15, c_1 = 0.002, a_1 = 0.01, a_2 = 0.02, k = 0.33 \end{aligned} \tag{20}$$

As $S_1(=50)$ is greater than computed value 33.3333 of $\dfrac{d_2}{ka_1}$ and 39.5 of $\dfrac{(d_1+e_2)}{c_1}$ in this case, the equilibrium point $E_1(50,0,0,0)$ is unstable and system (1) has two planar equilibrium points, say $E_2(39.5, 0.9545, 0, 0)$ and $E_3(33.3333,\ 0,\ 0.3333,\ 0)$. The equilibrium point $E_3(33.3333,\ 0,\ 0.3333,\ 0)$ is found to be stable as it satisfies the condition (12). The equilibrium point $E_2(39.5, 0.9545, 0, 0)$ is found to be unstable as it satisfies the condition (11). The solution trajectories in phase space S_1 - I_1 - S_2 are drawn in Fig. 5 for different initial values. All these trajectories approach to $E_3(33.3333,\ 0,\ 0.3333,\ 0)$. Convergence of solution trajectory to equilibrium $E_3(33.3333,\ 0,\ 0.3333,\ 0)$ with initial value near $E_2(39.5, 0.9545, 0, 0)$ also confirms the instability of $E_2(39.5, 0.9545, 0, 0)$.

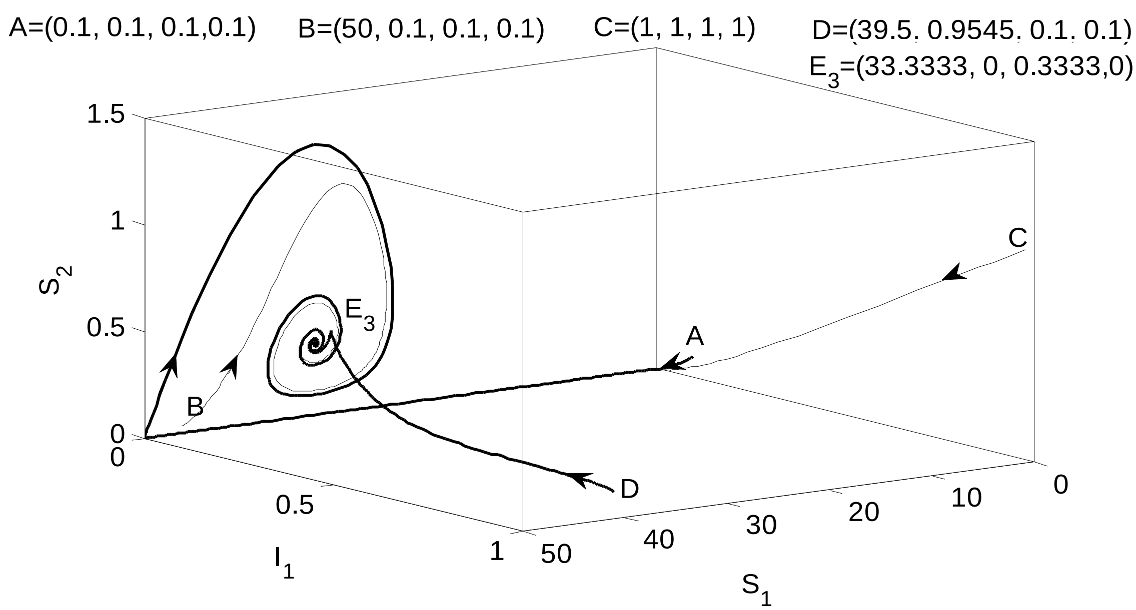

Fig. 5: Phase plot depicting the stability of E_3 and instability of E_2 in $S_1 - I_1 - S_2$ space for dataset (20)

7. Discussion and Conclusion

In this paper on predator-prey system, a disease transmission model, incorporating provisions for harvesting of prey have been discussed and analyzed. From the infected prey, the disease is spreading to susceptible prey and consequently to the predator species. The predator-prey species are compartmentalized into susceptible and infected classes. Recovery from disease is not considered for both prey and the predator species. Consequently, SI model is considered for both the species. SI model system admits four boundary equilibrium points and one non-zero interior equilibrium point. The dynamic behavior of the system around each equilibrium point has been studied and threshold values for basic reproduction numbers R_0 and R_1 are computed. Conditions for the existence and stability of disease free prey predator system are obtained. Conditions for endemic disease in prey species are discussed. Numerical Simulations are carried out to confirm the results obtained analytically.

It has been observed that the predator population becomes extinct and the disease dies out from the prey population for sufficiently small feeding efficiency of predator when the basic reproduction number R_0 in prey population is below the threshold. If the basic reproduction number R_0 in the prey population is above the threshold, the disease in prey population approaches endemic level and predator population becomes extinct. Disease in prey may be responsible for elimination of predator.

In case the feeding efficiency of predator population is sufficiently high so that $K_1 > \frac{rd_2}{(r-e_1)ka_1}\ (r \neq e_1)$ then the predator population persists. When the basic reproduction number R_1 is below the threshold, then the disease dies out. The prey and predator population both go to their persistent equilibrium values and the equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ is stabilized. The difference in death rates of infected and susceptible species may be responsible for instability of the equilibrium point $E^*(S_1^*, I_1^*, S_2^*, I_2^*)$. Therefore, the disease may not be endemic in prey predator system.

It is observed that harvesting of infected prey helps in making the system disease free where as harvesting of susceptible prey has adverse effect on it. Thus with an appropriate harvesting of susceptible and infected prey, disease free equilibrium point $E_3(\hat{S}_1, 0, \hat{S}_2, 0)$ can be maintained at desired level. It is also observed that with harvesting of infected prey only, size of prey and predator population is not affected where as removal of susceptible prey will not affect the prey level but will reduce the predator level in disease free equilibrium state. Removal of both susceptible and infected prey will make the system free from disease. At the same time it will destabilize the predator free equilibrium point. If $r < e_1$, i.e if intrinsic growth rate of susceptible prey is less than that of its harvesting from system then both the species will extinct from system.

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