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Distributional canonical sine transform by Using Gelfand-shilov technique

S.B.Chavhan

Department of Mathematics & Statistics,
Yeshwant Mahavidyalaya, Nanded-431602 (India)

ABSTRACT:
In this paper discussed the results on the topological structure of some of the S-type spaces of distributional canonical sine transform.

Keywords: Generalized function, testing function space, canonical sine transform

1 INTRODUCTION:

Extension of some transforms to generalized functions have been done time to time and their properties have been studies by various mathematicians,[3] [4],[5].Using Gelfand-shilov technique testing function spaces are defined [1],[2]. Notation and terminology of this paper is as per [6],[7]. The plan of this paper is as follows. Section 2 gives the different S-type testing spaces. Sections 3 discuss some bounded operations in S-type space. Section 4 prove important properties of S_γ^α space .

2 Different S-type testing function spaces: In this section we have defined s-type testing function spaces by imposing conditions not only on the decreases of the fundamental functions at infinity, but also on the growth of their derivatives as the order of derivative increases. Clearly, S^a space will be extension of testing function space D, so that these spaces have been successfully, used in pseudo differential operator theory. The open set $(R_+)\times(R_+)$ [first quadrant] and $(R_-)\times(R_+)$ [Second quadrant] are denoted I_1 and I_2 respectively. Unless and otherwise specified $A, B, \gamma, \beta \geq 0$. The letters, k, l means 0,1,2……

2.1 The space S_γ^a: It is given by

$$S_\gamma^a = \left\{\phi : \phi \in E_+ / \sigma_{l,k}\phi(t) = \sup_{I_1} \left|t^l . D_t^k \phi(t)\right| \leq C_k A^l . l^{l\gamma}\right\} \qquad (2.1)$$

The constant C_k and A depend on ϕ.

2.2 The space $S^{a,\beta}$:

$$S^{a,\beta} = \left\{\phi . \phi \in E_+ / \rho_{l,k}\phi(t) = \sup\left|t^l D_t^K \phi(t)\right| \leq C_l B^k k^{k\beta}\right\} \quad ..(2.2)$$

The constants C_l and B depend on ϕ.

2.3 The space $S_\gamma^{a,\beta}$ **:**This space is formed by combining the condition (2.1) and (2.2)

$$S_\gamma^{a,\beta} = \{\phi : \phi \in E_+ / \xi_{l,k}\phi(t) = \sup_{I_1} \left|t^l D_t^k \phi(t)\right| \leq C A^l l^{l\gamma} B^k k^{k,\beta}\} ..(2.3)$$

$l, k = 0,1,2...$ Where A,B,C depend on ϕ. In next we have introduced subspaces of each of the above space that are used in defining the inductive limits of these spaces.

2.4 The space $S_{\gamma,m}^a$ **:**It is defined as,

$$S^{a}_{\gamma,m} = \{\phi : \phi \in E_{+} / \sigma_{l,k}\phi(t) = \sup_{I_1} |t^l D_t^k \phi(t)| \le C_k (m+\mu)^l l^{l\gamma}\} \quad \text{.. (2.4)}$$

For any $\mu > 0$ where m is the constant, depending on the function ϕ.

2.5The space $S^{a,\beta,n}$: This space is given by

$$S^{a,\beta,n} = \{\phi : \phi \in E_{+} / \rho_{l,k}\phi(t) = \sup_{I_1} |t^l D_t^k \phi(t)| \le C_{l,\delta} (n+\delta)^k k^{k\beta}\} \quad (2.5)$$

For any $\delta > 0$ where n the constant is depends on the function ϕ.

2.6 The space $S^{a,\beta,n}_{\gamma,m}$: This space is defined by combining the conditions in (2.4) and (2.5).

$$S^{a,\beta,n}_{\gamma,m} = \{\phi : \phi \in E_{+} / \xi_{l,k}\phi(t) = \sup_{I_1} |t^l D_t^k \phi(t)| \le C_{\mu\delta} (m+\mu)^l (n+\delta)^k . l^{l\gamma} k^{k\beta}\} \quad (2.6)$$

For any $\mu > 0, \delta > 0$ and for given m > 0, n > 0 unless specified otherwise the space introduced in (2.1) through (2.6) will henceforth be consider equipped with their natural, Hausdoff, locally convex topologies to be denoted respectively by,

$$T^{a}_{\gamma}, T^{a,\beta}, T^{a,\beta}_{\gamma}, T^{a}_{\gamma,m}, T^{a,\beta,n}, T^{a,\beta,n}_{\gamma,m}$$

These topologies are respectively generalized by the total families of seminorms.

$$\{\sigma_{l,k}\}, \{\rho_{l,k}\}, \{\xi_{l,k}\}, \{\sigma_{l,k}\}, \{\rho_{l,k}\} \text{ and } \{\xi_{l,k}\}$$

3. Some Bounded operations in S-type spaces: This section is devoted to the study of different types of linear operators, namely, shifting operator, differentiation operator, scaling operator, in the $S^{a,\beta}_{\gamma}$ space. These operators are found to be bounded (continuous also) in the $S^{a,\beta}_{\gamma}$.

Proposition 3.1: If $\phi(t) \in S^{a,\beta}_{\gamma}$ and λ is fixed real number then $\phi(t+\lambda) \in C^{a,\beta}_{\gamma}, t+\lambda > 0$

Proof: Consider,

$$\xi_{lk}\phi(t+\lambda) = \sup_{I_1} |t^l D_t^k \phi(t+\lambda)|$$

$$\xi_{lk}\phi(t+\lambda) = \sup_{I_1} |(t' - \lambda)^l D_t^k \phi(t')|$$

where $t' = t + \lambda$

$$\le C A^l l^{l\gamma} B^k k^{k\beta}$$

thus $\phi(t+\lambda) \in S^{a,\beta}_{\gamma}$, for $t+\lambda > 0$.

Proposition 3.2: The translation (shifting) operator $T : \phi(t) \to \phi(t+\lambda)$ is a topological automorphism on $S^{a,\beta}_{\gamma}$ for $t+\lambda > 0$.

Proposition 3.3: If $\phi(t) \in S^{a,\beta}_{\gamma}$ and $\rho > 0$ strictly positive number then $\phi(\rho t) \in S^{a,\beta}_{\gamma}$

Proof: Consider

$$\xi_{l,k}\phi(\rho t) = \sup_{I_1} |t^l D_t^k \phi(\rho t)|$$

$$= \sup_{I_1} \left| \left(\frac{T}{\rho}\right)^l D_t^k \phi(T) \right|$$

$$= C_l \sup_{I_1} |T^l D_T^k \phi(T)|$$

Where C_l is constant depending on 'ρ'

$$\le C_1 C_2 A^l l^{l\gamma} B^k . k^{k\beta}$$

$$\le C A^l . l^{l.y} . B^k K^{k\beta},$$

Where $C = C_1C_2$ Thus $\phi(\rho t) \in S_\gamma^{a,\beta}$ for $\rho > 0$

Proposition 3.4: If $\rho > 0$ and $\phi(t) \in S_\gamma^{a,\beta}$ then the scaling operator.

$R : S_\gamma^{a,\beta} \to S_\gamma^{a,\beta}$ defined $R\phi = \psi$

Where $\psi(t) = \phi(\rho t)$ is a topological automorphism.

Proposition 3.5: The operator $\phi(t) \to D_t\phi(t)$ is defined on the space $S_\gamma^{a,\beta}$ and transform this space into itself

Proof: Let $\phi(t) \in S_\gamma^{a,\beta}$,if $D_t\phi(t) = \psi(t)$ we have,

$$\xi_{l,k}(\psi) = \sup_{I_1} \left| t^l D_t^k \psi(t) \right|$$

$$= \sup_{I_1} \left| t^l D_t^k D_t(t) \right|$$

$$= \sup_{I_1} \left| t^l D_t^{k+1}(t) \right|$$

$$\leq C A^l l^{l\gamma} B^{(k+1)} (k+1)^{(k+1)\beta}$$

$$\therefore \psi(t) \in S_\gamma^{a,\beta}$$

4. Topological properties of S_γ^a - space:

This section is devoted to discuss the result on the topological structures of some of the spaces and the results exhibiting their relationship. Then attention is also paid to be strict inductive limits of some of these spaces.

Theorem 4.1: $\left(S_\gamma^a, T_\gamma^a\right)$ is a Frechet space

Proof: As the family A_γ^a of seminorms $\{\delta_{l,k}\}_{l,k=0}^{\infty}$ generating T_γ^a is countable it, suffices to prove the completeness of the space $\left(S_\gamma^a, T_\gamma^a\right)$.

Let us consider a Cauchy sequence $\{\phi_n\}$ in S_γ^a. Hence for a given $\in > 0$ there exist

an $N = N_{l,k}$ such that for $m, n \geq N$

$$\delta_{l,k}(\phi_m - \phi_n) = \sup_{I_1} \left| t^l D_t^k (\phi_m - \phi_n) \right| < \in \quad (4.1)$$

In particular for $l = k = 0, m, n \geq N$

$\sup_{I_1} \left| \phi_m(t) - \phi_n(t) \right| < \in$ (4.2) Consequently for fixed t in I_1 $\{\phi(t)\}$ is a numerical Cauchy sequence. Let $\phi(t)$ be the point wise limit of $\{\phi_m(t)\}$ using (4.2) we can easily deduce that $\{\phi_m(t)\}$ converges to ϕ uniformly on I_1. Thus ϕ is continuous moreover, repeated use of (4.1) for different values *l,k*, yields that ϕ is smooth i.e. $\phi \in E_+$ further from,(4.1) We get, $\delta_{l,k}(\phi_m) \leq \delta_{l,k}(\phi_N) + \in \quad \forall m \geq n$

$$\leq C_k A^l l^{l,y} + E$$

taking $m \to \infty$ and $\in$ is arbitrary we get,

$$\delta_{l,k}(\phi) = \sup_{I_1} \left| t^l D_t^k \phi(t) \right|$$

$$\leq C_k A^l l^{l\gamma}$$

Hence $\phi \in S_\gamma^a$ and it is the T_γ^a limit of ϕ_m by (4.1).

This proves the completeness of S_γ^a and $\left(S_\gamma^a, T_\gamma^a\right)$ is a Frechet space.

Proposition 4.2: The space $D(I_1)$ is a subspace of S_γ^a such that the injection mapping from $D(I_1)$ to S_γ^a is continuous i.e $T_\gamma{}^a / D(I_1) \subset T(I_1)$.

Proof: For $\phi(t) \in D(I_1)$, Set $L = \sup_{I_1} \{t : t \in \sup p\phi\}$

and $C_k = \sup_{I_1} \left| D_t^k \phi(t) \right|$, then

$$\delta_{l,k}\phi(t) = \sup_{I_1} \left| t^l D_t^k \phi(t) \right|,$$

$$\leq C_k L^l$$

$$\leq C_k L^l \frac{l^{l\gamma}}{l^{l\gamma}} \cdot \frac{A^l}{A^l}$$

$$= C_k \left[\frac{L}{A l^\gamma} \right]^l . A l^{l\gamma} \qquad (4.3)$$

since $\frac{L}{A l^\gamma} \leq 1$ iff $l \geq \left(\frac{L}{A}\right)^{\frac{1}{\gamma}}$ define

$$l_0 = \left[\left(\frac{L}{A}\right)^{\frac{1}{\gamma}} \right] + 1$$

where $[t]$ denotes the Gaussian symbol, that is greatest integer not exceeding t. Therefore $l > l_0$ we have

$$\delta_{l,k}\phi(t) \leq C_k A^l . l^{l\gamma} \qquad (4.4)$$

if $l \leq l_0$ let us write,

$$C = \max \left\{ \frac{L}{A} \left[\frac{L}{A l_2^\gamma} \right]^2, \ldots\ldots\ldots\ldots \left[\frac{L}{A l_0^\gamma} \right]^{l_0} \right\}.$$

Then again from (4.2)

$$\delta_{l,k}\phi(t) \leq C C_k A^l . l^{l\gamma} \qquad (4.5)$$

Hence the inequalities (4.4) and (4.5) together yield,

$$\delta_{l,k}\phi(t) \leq C C_k A^l . l^{l\gamma}$$

$$\leq C_k^{'} A^l . l^{l\gamma} \quad \forall\ l \geq 0$$

Implying that $\phi \in S_\gamma^a$.

Consequently, $D(I_1) \in S_\gamma^a$

To prove that continuity of the injective map, consider a sequence

$\{\phi_n\}$ in $\mathrm{D}(I_1)$, converging to zero.

Then

$$\delta_{l,k}\{\phi_n\} = \sup_{I_1} \left|t^l D_t^k \phi_n(t)\right| \qquad (4.6)$$

Now we find a compact subset Q_m (say) of I_1 Such that if $\phi_n \in D(Q_m)$ for each $n \geq 1$ and $\phi_n \to 0$ in $D(Q_m)$

if $P = \sup_{Q_m} \left|t^l\right|$ then from (4.6)

$$\delta_{l,k}\{\phi_n\} \leq P \sup_{Q_m} \left|D_t^k \phi_n(t)\right| \to 0 \quad \text{as } n \to \infty$$

Hence $\phi_n \to 0$ in C_γ^a and therefore $T_\gamma^{\ a} / D(I_1) \subset T(I_1)$.

Proposition 4.3: If $m_1 < m_2$ then $S_{\gamma,m_1}^a \subset S_{\gamma,m_2}^a$. The topology of $S_{\gamma,m_1}^{a,}$ is equivalent to the topology induced on $S_{\gamma,m_1}^{a,}$ by $S_{\gamma,m_2}^{a,}$

$$i.e \; T_{\gamma,m_1}^{\ a} \sim T_{\gamma,m_2}^{\ a} / S_{\gamma,m_1}^a$$

Proof: For $\phi \in S_{\gamma,m_1}^{a,}$ and

$$\delta_{l,k}(\phi) \leq C_{k,\mu}(m_1+\mu)^l \, l^{l\gamma}$$

$$\leq C_{k,\mu}(m_2+\mu)^l \, l^{l\gamma} \qquad \text{thus,}$$

$S_{\gamma,m_1}^a \subset S_{\gamma,m_2}^a$

The second part is clearly from the definition of topologies of these spaces. The space $S_{\gamma,}^{a,}$ can be expressed as union of countably normed spaces.

Proposition 4.4: $S_\gamma^a = \bigcup_{i=1}^{\infty} S_{\gamma,m_1}^a$ and if the space S_γ^a is equipped with strict inductive limit topology $S_{a,m}$ defined by injective map from S_{γ,m_1}^a to S_γ^a then the sequence $\{\phi_n\}$ in S_γ^a converges to zero.

Proof: we show that $S_\gamma^a = \bigcup_{i=1}^{\infty} S_{\gamma.m_1}^a$

Clearly $\bigcup_{i=1}^{\infty} S_{\gamma.m_1}^a \subset S_k^a$ for proving the other inclusion, let $\phi \in C_\gamma^a$ then

$$\delta_{l,k}(\phi) = \sup_{I_1} \left|t^l D_t^k \phi_n(t)\right|$$

$$\leq C_k A^l l^{l\gamma}, \qquad (4.7)$$

where A is some positive constant, choose an integer $m = m_A$ and $\mu = 0$ such that $C_k A^l \leq C_k (m+\mu)^l$. .

Then (4.7) we get $\phi \in S_{\gamma,m_1}^a$ implying that $S_\gamma^a = \bigcup_{i=1}^{\infty} S_{\gamma,m_1}^a$

Proposition 4.5: If $\gamma_1 < \gamma_2$ and $\beta_1 < \beta_2$ then $S_{\gamma_1}^{a,\beta_1} \subset S_{\gamma_2}^{a,\beta_2}$ and the topology of $S_{\gamma_i}^{a,\beta_i}$ is equivalent to the topology induced on $S_{\gamma_1}^{a,\beta_1}$ by $S_{\gamma_2}^{a,\beta_2}$.

Proof: Let $\phi \in S_{\gamma_i}^{a,\beta_i}$

$$\xi_{l,k}(\phi) = \sup_{I_1} \left|t^l D_t^k \phi(t)\right|$$

$$\leq CA^l l^{l\gamma_1} . B^k k^{k\beta_1}$$

$$\leq CA^l l^{l\gamma_2} . B^k k^{k\beta_2} \qquad \text{where } l,k = 0,1,2,3$$

Hence $\phi \in S_{\gamma_2}^{a,\beta_2}$. Consequently, $S_{\gamma_1}^{a,\beta_1} \subset S_{\gamma_2}^{a,\beta_2}$. The topology of $S_{\gamma_i}^{a,\beta_i}$

Is equivalent to the topology $T_{\gamma_2}^{a,\beta_2} / S_{\gamma_2}^{a,\beta_2}$

it is clear from the definition of topologies of these spaces.

Proposition 4.6: $S^a = \underset{\gamma_i \beta_i = 1}{\overset{\infty}{U}} S_{\gamma_i}^{a,\beta_i}$ and if the space C^a is equipped with the strict S^a inductive limit topology defined by the injective maps from $S_{\gamma_i}^{a,\beta_i}$ to S^a then the sequence $\{\phi_n\}$ in S^a converges to zero iff $\{\phi_n\}$ is contained in some $S_{\gamma_i}^{a,\beta_i}$ and converges to zero.

Proof: The above theorem is an immediate consequence of last theorem

$$S^a = \underset{\gamma_i \beta_i = 1}{\overset{\infty}{U}} S_{\gamma_i}^{a,\beta_i}$$

Clearly $\underset{\gamma_i \beta_i = 1}{\overset{\infty}{U}} S_{\gamma_i}^{a,\beta_i} \subset S^a$

For proving other inclusion, let $\phi \in S^a$ then

$$\eta_{l,k}(\phi) = \sup_{I_1} \left| t^l D_t^k \phi(t) \right|,$$

is bounded by some number. We can choose integers γ_m and β_m such that

$$\eta_{l,k}(\phi) \leq C A^l l^{l\gamma} B^k, k^{k\beta}$$

$\therefore \phi \in S_{\gamma_i}^{a,\beta_i}$ for some integer γ_i and β_i

Hence $S^a \subset \underset{\gamma_i \beta_i = 1}{\overset{\infty}{U}} S_{\gamma_i}^{a,\beta_i}$

Thus $S^a = \underset{\gamma_i \beta_i = 1}{\overset{\infty}{U}} S_{\gamma_i}^{a,\beta_i}$

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